

# Chabauty limits of algebraic groups acting on trees

## The quasi-split case

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### Abstract

Given a locally finite leafless tree  $T$ , various algebraic groups over local fields might appear as closed subgroups of  $\text{Aut}(T)$ . We show that the set of closed cocompact subgroups of  $\text{Aut}(T)$  that are isomorphic to a quasi-split simple algebraic group is a closed subset of the Chabauty space of  $\text{Aut}(T)$ . This is done via a study of the integral Bruhat–Tits model of  $\text{SL}_2$  and  $\text{SU}_3^{L/K}$ , that we carry on over arbitrary local fields, without any restriction on the (residue) characteristic. In particular, we show that in residue characteristic 2, the Tits index of simple algebraic subgroups of  $\text{Aut}(T)$  is not always preserved under Chabauty limits.

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# 1 Introduction

*Ta vague monte avec la rumeur d'un prodige  
C'est ici ta limite. Arrête-toi, te dis-je.  
(Victor Hugo, L'année terrible, 1872)*

According to well-known rigidity results of J. Tits (see [Tit74, Theorem 5.8], together with [Tit86, Théorème 2] or [Wei09, Theorem 27.6]), a Bruhat–Tits building of rank  $\geq 2$  determines uniquely the simple algebraic group and the underlying ground field to which it is associated. In particular, two simply connected absolutely simple algebraic groups over local fields of relative rank  $\geq 2$  have isomorphic Bruhat–Tits buildings if and only if they are isomorphic as locally compact groups. This contrasts drastically with the rank 1 case, where infinitely many pairwise non-isomorphic simple algebraic groups of relative rank 1 can have the same Bruhat–Tits tree. Therefore, given a locally finite leafless tree  $T$ , the set  $\mathbf{Sub}(\mathrm{Aut}(T))$  of closed subgroups of the locally compact group  $\mathrm{Aut}(T)$  may contain infinitely many pairwise non-isomorphic algebraic groups. For example, the Bruhat–Tits tree of the split group  $\mathrm{SL}_2(K)$  is completely determined by the order of the residue field of  $K$ , while the isomorphism type of  $\mathrm{SL}_2(K)$  depends on the isomorphism type of the local field  $K$ . Since  $\mathbf{Sub}(\mathrm{Aut}(T))$  carries a natural compact Hausdorff topology, namely the Chabauty topology, we are naturally led to the following question: *what are the Chabauty limits of algebraic groups in  $\mathbf{Sub}(\mathrm{Aut}(T))$ ?* The goal of this paper is to initiate the study of that problem. In particular, we provide a complete solution in the case of quasi-split groups.

In order to be more precise, for  $T$  a tree, let us define a *topologically simple algebraic group acting on  $T$*  to be a locally compact group isomorphic to  $H(K)/Z$ , where  $K$  is a local field,  $H$  is an absolutely simple, simply connected, algebraic group over  $K$  of relative rank 1 whose Bruhat–Tits tree is isomorphic to  $T$ , and  $Z$  is the center of  $H(K)$ .

The first thing to observe is that, given a topologically simple algebraic group  $G$  acting on  $T$ , the action homomorphism  $G \rightarrow \mathrm{Aut}(T)$  is not canonical, but depends on some choices. There is however a natural way to resolve this issue of canonicity, explained in [CR16]. Following that paper, we shall denote by  $\mathcal{S}_T$  the space of (topological) isomorphism classes of topologically simple closed subgroups of  $\mathrm{Aut}(T)$  acting 2-transitively on the set of ends. According to [CR16, Theorem 1.2], the space  $\mathcal{S}_T$  endowed with the quotient topology induced from the Chabauty space  $\mathbf{Sub}(\mathrm{Aut}(T))$  is compact Hausdorff.

We can therefore reformulate the question mentioned above as follows. Let  $\mathcal{S}_T^{\mathrm{alg}}$  be the set of isomorphism classes of topologically simple algebraic groups acting on  $T$ . *What are the accumulation points in  $\mathcal{S}_T$  of the elements of  $\mathcal{S}_T^{\mathrm{alg}}$ ?* It seems reasonable to conjecture that  $\mathcal{S}_T^{\mathrm{alg}}$  is closed in  $\mathcal{S}_T$ . Our main theorem is a partial result in this direction.

**Theorem 1.1.** *Let  $T$  be a locally finite leafless tree, and let  $\mathcal{S}_T^{\mathrm{qs-alg}}$  be the set of isomorphism classes of topologically simple algebraic groups acting on  $T$  that are furthermore quasi-split. Then  $\mathcal{S}_T^{\mathrm{qs-alg}}$  is closed in  $\mathcal{S}_T$ .*

As recalled in Section 2.1, the classification of the simple algebraic groups over local fields implies that absolutely simple, simply connected, quasi-split algebraic groups over  $K$  of relative rank 1 are of the form  $\mathrm{SL}_2(K)$  or  $\mathrm{SU}_3^{L/K}(K)$  (see Lemma 2.3). So that in effect, the main goal of the paper is only to dispose of those two “types” of groups.

Since the Bruhat–Tits tree of  $\mathrm{SL}_2(K)$  or  $\mathrm{SU}_3^{L/K}(K)$  for  $L$  a ramified extension of  $K$  (respectively  $\mathrm{SU}_3^{L/K}(K)$  for  $L$  an unramified extension of  $K$ ) is isomorphic to the  $(p^n + 1)$ -regular tree (respectively the semiregular tree of bidegree  $(p^{3n} + 1; p^n + 1)$ ), where  $p^n$  is the order of the residue field of  $K$ , the space  $\mathcal{S}_T^{\mathrm{qs-alg}}$  is empty unless  $T$  is one of those trees.

It should also be noted that for some trees  $T$ , every algebraic group having  $T$  as Bruhat–Tits tree is actually quasi-split. According to the classification tables in [Tit79, 4.2 and 4.3], this is the case if and only if  $T$  is the regular tree of degree  $p + 1$  or the semiregular tree of bidegree  $(p^{3n} + 1; p^n + 1)$ . Combining this observation with Theorem 1.1, we get the following corollary.

**Corollary 1.2.** *Let  $p$  be a prime number, and let  $T$  be the  $(p+1)$ -regular tree, or the  $(p^{3n}+1; p^n+1)$ -semiregular tree. Then the set  $\mathcal{S}_T^{\text{alg}}$  coincides with  $\mathcal{S}_T^{\text{qs-alg}}$ , so that it is closed in  $\mathcal{S}_T$ .*

In fact, we give an explicit description of the topological space  $\mathcal{S}_T^{\text{qs-alg}}$ . To achieve it, we proceed in two steps. We first describe the space  $\mathcal{L}$  of quadratic pairs of local fields (as defined in Definition 5.3), and the purpose of Section 5.1 is to give an explicit description of  $\mathcal{L}$ , which appears in Proposition 5.12. The process is a bit lengthy, but only uses elementary facts about local fields. In a second step, we show in the proof of Theorem 1.3 that the map

$$\mathcal{L} \rightarrow \mathcal{S}_T: (K, L) \mapsto \hat{G}_{(K, L)}$$

is a homeomorphism onto its image (see Definition 6.1 and Proposition 6.4 for the definition of this map). Note that we make an abuse of notation: we represent a point in  $\mathcal{S}_T$ , which is an isomorphism class, by a representative of that class. This abuse should not cause any confusion, and will simplify notations throughout the rest of the paper.

To ease the statement of the explicit form of the main theorem, let us introduce some terminology. Recall that a countable totally disconnected topological space  $X$  is classified by two invariants (see [MS20, Théorème 1]). More precisely, let  $\hat{\mathbf{N}}$  be the one point compactification of  $\mathbf{N}$  (or in other words, a topological space homeomorphic to  $\{1, \frac{1}{2}, \frac{1}{3}, \dots, 0\} \subset \mathbf{R}$ ). If  $X^{(k)}$  is the last non-empty Cantor-Bendixson derivative of  $X$ , and if  $X^{(k)}$  has  $n$  connected components, then  $X$  is homeomorphic to  $\hat{\mathbf{N}}^k \times \{1, \dots, n\}$ .

**Theorem 1.3.**

1. *Let  $p$  be an odd prime number, and let  $T$  be the  $(p^n + 1)$ -regular tree. Then there exists a homeomorphism  $f: \hat{\mathbf{N}} \times \{1, 2\} \rightarrow \mathcal{S}_T^{\text{qs-alg}}$  such that*

$$\begin{aligned} f(\hat{\mathbf{N}} \times \{1\}) &= \{\text{SL}_2(K)/Z \mid \overline{K} \cong \mathbf{F}_{p^n}\} \\ f((\infty, 1)) &= \text{SL}_2(\mathbf{F}_{p^n}((X)))/Z \\ f(\hat{\mathbf{N}} \times \{2\}) &= \{\text{SU}_3^{L/K}(K)/Z \mid \overline{K} \cong \mathbf{F}_{p^n} \text{ and } L \text{ is (separable) quadratic ramified}\} \\ f((\infty, 2)) &= \text{SU}_3^{L_0/\mathbf{F}_{p^n}((X))}(\mathbf{F}_{p^n}((X)))/Z \end{aligned}$$

where  $\infty$  denotes the accumulation point of  $\hat{\mathbf{N}}$ ,  $\overline{K}$  denotes the residue field of  $K$ , and  $L_0$  is any (separable) quadratic ramified extension of  $\mathbf{F}_{p^n}((X))$ .

2. *Let  $T$  be the  $(2^n + 1)$ -regular tree. Then  $\mathcal{S}_T^{\text{qs-alg}}$  is homeomorphic to  $\hat{\mathbf{N}}^2$ . More precisely,*

$$\begin{aligned} \mathcal{S}_T^{\text{qs-alg}} &= \{\text{SL}_2(K)/Z \mid \overline{K} \cong \mathbf{F}_{2^n}\} \\ &\cup \{\text{SU}_3^{L/K}(K)/Z \mid \overline{K} \cong \mathbf{F}_{2^n} \text{ and } L \text{ is separable quadratic ramified}\} \end{aligned}$$

The first Cantor-Bendixson derivative of  $\mathcal{S}_T^{\text{qs-alg}}$  is

$$\{\text{SU}_3^{L/\mathbf{F}_{2^n}((X))}(\mathbf{F}_{2^n}((X)))/Z \mid L \text{ is separable quadratic ramified}\} \cup \{\text{SL}_2(\mathbf{F}_{2^n}((X)))/Z\}$$

while its second Cantor-Bendixson derivative contains the single element  $\text{SL}_2(\mathbf{F}_{2^n}((X)))/Z$ .

3. *Let  $p$  be any prime number, and let  $T$  be the  $(p^{3n} + 1; p^n + 1)$ -semiregular tree. Then  $\mathcal{S}_T^{\text{qs-alg}}$  is homeomorphic to  $\hat{\mathbf{N}}$ . More precisely,*

$$\mathcal{S}_T^{\text{qs-alg}} = \{\text{SU}_3^{L/K}(K)/Z \mid \overline{K} \cong \mathbf{F}_{p^n} \text{ and } L \text{ is (separable) quadratic unramified}\}$$

Furthermore, the accumulation point of  $\mathcal{S}_T^{\text{qs-alg}}$  is  $\text{SU}_3^{L/\mathbf{F}_{p^n}((X))}(\mathbf{F}_{p^n}((X)))/Z$ , where  $L$  is the (separable) quadratic unramified extension of  $\mathbf{F}_{p^n}((X))$ .

As one can see from Theorem 1.3, we face a more complex situation in residue characteristic 2. Indeed, that statement implies that the split group  $SL_2(\mathbf{F}_{2^n}((X)))/Z$  is a limit of unitary groups, thereby illustrating the fact that the Tits index need not be preserved under Chabauty limits in residue characteristic 2. In other words, the map associating to an isomorphism class in  $\mathcal{S}_T^{\text{alg}}$  its Tits index is not continuous.

Since the map  $\mathcal{L} \rightarrow \mathcal{S}_T$  is a homeomorphism onto its image, the complexity of the residue characteristic 2 case should already be visible at the level of the space  $\mathcal{L}$  of quadratic pairs of local fields. And indeed, Proposition 5.12 reflects this fact. The specific features of Chabauty limits in residue characteristic 2 highlight the complexity of the aforementioned conjecture, which will be addressed in full generality in a forthcoming paper, but with different methods.

The strategy to prove our results is the same for all algebraic groups under consideration (i.e.  $SL_2$  or  $SU_3$ ). Let us outline it in the  $SL_2$  case (our notational conventions for local fields are spelled out at the beginning of Section 2.1).

1. In Definition 3.10, we recall the definition of the Bruhat–Tits tree:

$$\mathcal{I} = SL_2(K) \times \mathbf{R}/\sim$$

2. In Definition 3.14, we define a pointed version (around 0) of the Bruhat–Tits tree:

$$\mathcal{I}_0 = SL_2(\mathcal{O}_K) \times \mathbf{R}/\sim_0$$

and in Lemma 3.16, we show that the homomorphism  $SL_2(\mathcal{O}_K) \rightarrow SL_2(K)$  induces an  $(SL_2(\mathcal{O}_K) \rightarrow SL_2(K))$ -equivariant bijection  $\mathcal{I}_0 \rightarrow \mathcal{I}$ .

3. In Definition 4.29, we define the ball around 0 of radius  $r$ :

$$B_0(r) = \{[(g, x)]_0 \in \mathcal{I}_0 \mid x \in [-\omega(\pi_K^r), \omega(\pi_K^r)] \subset \mathbf{R}, g \in SL_2(\mathcal{O}_K)\}$$

4. In Definition 4.9, we define a local version (around 0 and of radius  $r$ ) of the Bruhat–Tits tree:

$$\mathcal{I}^{0,r} = SL_2(\mathcal{O}_K/\mathfrak{m}_K^r) \times [-\omega(\pi_K^r), \omega(\pi_K^r)]/\sim_{0,r}$$

and we show in Theorem 4.32 that the homomorphism  $SL_2(\mathcal{O}_K) \rightarrow SL_2(\mathcal{O}_K/\mathfrak{m}_K^r)$  induces an  $(SL_2(\mathcal{O}_K) \rightarrow SL_2(\mathcal{O}_K/\mathfrak{m}_K^r))$ -equivariant bijection  $B_0(r) \rightarrow \mathcal{I}^{0,r}$ .

5. Following an idea dating back to M. Krasner (see [Del84] for references, this idea is also used in e.g. [Kaz86]), we define a metric  $d$  on the space  $\mathcal{K}$  of (isomorphism classes of) local fields by declaring that for  $r \in \mathbf{N}$  and  $K_1, K_2 \in \mathcal{K}$ ,  $d(K_1; K_2) \leq \frac{1}{2^r}$  if and only if  $\mathcal{O}_{K_1}/\mathfrak{m}_{K_1}^r \cong \mathcal{O}_{K_2}/\mathfrak{m}_{K_2}^r$  (see Lemma 5.7). We observe in Proposition 5.12 that the space  $\mathcal{K}_{p^n}$  of (isomorphism classes of) local fields having residue field  $\mathbf{F}_{p^n}$  is homeomorphic to  $\hat{\mathbf{N}}$ .
6. Points 1 to 4 imply that if  $K_1$  and  $K_2$  are close to each other in  $\mathcal{K}_{p^n}$ , then  $SL_2(\mathcal{O}_{K_1})$  and  $SL_2(\mathcal{O}_{K_2})$  are close to each other in the Chabauty space of  $\text{Aut}(T_{p^n+1})$  (where  $T_{p^n+1}$  is the  $(p^n + 1)$ -regular tree). Indeed, up to isomorphism, they act in the same way on a large ball centred at 0. This is the key step in the proof of Theorem 6.5.
7. We are then able to conclude effortlessly, using a rigidity argument, that the map  $\mathcal{K}_{p^n} \rightarrow \mathcal{S}_{T_{p^n+1}}^{\text{alg}} : K \mapsto SL_2(K)/Z$  is a homeomorphism onto its image.

A key tool to implement our strategy is the existence of good functors from  $\mathcal{O}_K$ -algebras (such as  $\mathcal{O}_K/\mathfrak{m}_K^r$ ) to groups (like  $SL_2(\mathcal{O}_K/\mathfrak{m}_K^r)$ ). The integral model provided by Bruhat–Tits theory plays the role of this good functor. In the  $SL_2$  case, this is just the algebraic group  $SL_2$  considered over  $\mathcal{O}_K$ . But a description of the integral model is not always so straightforward, and an important feature of this article is an explicit computation of Bruhat–Tits models for  $SU_3^{L/K}$ , especially in the more delicate case when the residue characteristic is 2 and  $L$  is ramified.

The complexity of the integral model of  $\mathrm{SU}_3^{L/K}$  when the residue characteristic is 2 and  $L$  is ramified also explains why we get a different behaviour for regular trees of degree  $2^n + 1$  in Theorem 1.3. As often in the theory of algebraic groups, the characteristic 2 case is more involved to work out (and in our situation, it is again because of the presence of orthogonal groups in characteristic 2 lurking in the background, see Remark 4.14), but as was strongly advocated by J. Tits, this case is also of great interest. Our results seem to be another illustration of this philosophy.

It also appears that studying convergence of groups isomorphic to  $\mathrm{SL}_2(D)/Z$  (where  $D$  is a finite dimensional central division algebra over a local field  $K$ ) can be done in parallel to the  $\mathrm{SL}_2(K)$  case. Hence we decided to treat this case as well in this paper. We stress that this is only an opportunistic choice, and that the other cases should be settled by first considering similar questions in arbitrary rank for quasi-split groups, and then by applying a descent method.

Nevertheless, thanks to this treatment, we get the following results as well.

**Theorem 1.4.** *Let  $T$  be a locally finite leafless tree, and let  $\mathcal{S}_T^{\mathrm{SL}_2(D)}$  be the set of isomorphism classes of topologically simple algebraic groups acting on  $T$  that are furthermore isomorphic to  $\mathrm{SL}_2(D)/Z$  for some central division algebra  $D$ . Then  $\mathcal{S}_T^{\mathrm{SL}_2(D)}$  is closed in  $\mathcal{S}_T$ .*

Hence, for the reasons explained before Corollary 1.2 and according to the tables in [Tit79, 4.2 and 4.3], we obtain the following strengthening of Corollary 1.2.

**Corollary 1.5.** *Let  $p$  be a prime number, and let  $T$  be the  $(p^n + 1)$ -regular tree where  $n$  is not divisible by 3, or the  $(p^{3n} + 1; p^n + 1)$ -semiregular tree. Then the set  $\mathcal{S}_T^{\mathrm{alg}}$  coincides with  $\mathcal{S}_T^{\mathrm{qs-alg}} \cup \mathcal{S}_T^{\mathrm{SL}_2(D)}$ , so that it is closed in  $\mathcal{S}_T$ .*

Again, just as for the quasi-split case, we are actually able to describe explicitly the topological space  $\mathcal{S}_T^{\mathrm{SL}_2(D)}$  and all the convergences in this space.

**Theorem 1.6.** *Let  $T$  be the  $(p^n + 1)$ -regular tree.*

1. *The topological space  $\mathcal{S}_T^{\mathrm{SL}_2(D)}$  is homeomorphic to  $\hat{\mathbf{N}} \times \{1, \dots, \lceil \frac{n+1}{2} \rceil\}$ . The first Cantor-Bendixson derivative of  $\mathcal{S}_T^{\mathrm{SL}_2(D)}$  is*

$$\{\mathrm{SL}_2(D)/Z \mid \overline{D} \cong \mathbf{F}_{p^n} \text{ and } D \text{ is of characteristic } p\}$$

2. *For  $i \in \mathbf{N}$ , let  $D_i$  (respectively  $D$ ) be a finite dimensional central division algebra over  $K_i$  (respectively  $K$ ) having residue field of cardinality  $p^n$ . Let  $d_i$  (respectively  $d$ ) be the degree of  $D_i$  (respectively  $D$ ), so that  $|\overline{K}_i|^{d_i} = p^n = |\overline{K}|^d$ , where  $\overline{K}_i$  (respectively  $\overline{K}$ ) denotes the residue field of  $K_i$  (respectively  $K$ ). Let  $r_i$  (respectively  $r$ ) be the Hasse invariant of  $D_i$  (respectively  $D$ ), as in Definition B.2. If  $(\mathrm{SL}_2(D_i))_{i \in \mathbf{N}}$  converges to  $\mathrm{SL}_2(D)$  in the Chabauty space  $\mathbf{Sub}(\mathrm{Aut}(T))$ , then for all  $i$  large enough,  $r_i = \pm r$  and  $d_i = d$ , so that  $|\overline{K}_i| = |\overline{K}|$  as well.*

We conclude this introduction by mentioning the recent work of M. de la Salle and R. Tessera [dlST15], who used independently closely related ideas in their study of the space of Bruhat–Tits buildings of type  $\tilde{A}_n$  (with  $n > 2$ ) endowed with the Gromov–Hausdorff topology.

## Acknowledgements

I gratefully thank an anonymous commenter to a question on the website MathOverflow (see [Stu]) for explaining how to extract from [Gro67] the form of Hensel’s Lemma we needed. We reproduced his comments for the proof of Theorem 4.25. We are also grateful to M. de la Salle and R. Tessera for their comments on this work, which led me to include the  $\mathrm{SL}_2(D)$  case. Furthermore, I warmly thank P.-E. Caprace and N. Radu for introducing me to this interesting topic, and for their enthusiasm about this work. The former also gave the slick argument to deduce Chabauty convergence of the whole group from Chabauty convergence of vertex stabilisers, while the latter also made the initial breakthrough by computing Chabauty limits in the  $\mathrm{SL}_2$  case.

## 2 Definitions of the algebraic groups under consideration

For the rest of the paper,  $K$  will denote a local field (all our local fields are assumed to be non-archimedean), and  $D$  will denote a finite dimensional central simple division algebra over  $K$ . Let us spell out our notational conventions for the objects associated with  $K$  (respectively  $D$ ): the ring of integers is denoted  $\mathcal{O}_K$  (respectively  $\mathcal{O}_D$ ), its maximal ideal by  $\mathfrak{m}_K$  (respectively  $\mathfrak{m}_D$ ), a uniformiser by  $\pi_K$  (respectively  $\pi_D$ ) and  $\overline{K}$  (respectively  $\overline{D}$ ) denotes the residue field. The valuation of  $K$  (respectively  $D$ ), and also its unique extension to any finite extension of  $K$ , is denoted by  $\omega$ . We use the notation  $\mathbf{Q}_{p^n}$  for the unique (up to isomorphism) unramified extension of  $\mathbf{Q}_p$  of degree  $n$ .

Also, in order to avoid the repetition of long lists of adjectives, in this section, by an algebraic group, we mean an absolutely simple, simply connected algebraic group over a local field.

### 2.1 Quasi-split groups of relative rank 1

As mentioned in the introduction, the Bruhat–Tits building of an algebraic group  $G$  is a tree if and only if  $G$  is of relative rank 1. Instead of giving the general definition of quasi-split algebraic groups, and then specialising to those that are of relative rank 1, we take a practical approach and give an explicit description of those groups, the result being that they are all of the form  $\mathrm{SL}_2$  or  $\mathrm{SU}_3$ . We begin by recalling the definition of  $\mathrm{SU}_3$ .

**Definition 2.1.** Let  $L$  be a separable quadratic extension of  $K$ , and let  $\sigma$  be the nontrivial element of  $\mathrm{Aut}(L/K)$ , whose action by conjugation on  $L$  is denoted  $x \mapsto \bar{x}$ . Consider the transposition along the anti-diagonal  ${}^S(\cdot): \mathrm{SL}_3(L) \rightarrow \mathrm{SL}_3(L): g \mapsto {}^Sg$ . More explicitly,  $({}^Sg)_{-j,-i} = g_{ij}$ , for  $i, j \in \{-1, 0, 1\}$ . Then we define

$$\mathrm{SU}_3^{L/K}(K) = \{g \in \mathrm{SL}_3(L) \mid {}^S\bar{g}^{-1} = g\}$$

We denote  $\mathrm{SU}_3^{L/K}$  (or simply  $\mathrm{SU}_3$  when the pair of field  $(K, L)$  is arbitrary or understood from the context) the corresponding algebraic group over  $K$ . Note that the equations  $\det(g) = 1$  and  ${}^S\bar{g}g = \mathrm{Id}$  (together with the embedding  $L \hookrightarrow M_2(K)$ ) realise  $\mathrm{SU}_3^{L/K}$  as a closed subspace of the affine space  $\mathbf{A}_K^n$  of dimension  $n = 4 \times 3^2$ . Using this, it is readily seen that  $\mathrm{SU}_3$  is an algebraic group over  $K$ .

**Remark 2.2.** The group  $\mathrm{SU}_3$  defined above is the special unitary group with respect to the following hermitian form of  $L^3$ :

$$((x_{-1}, x_0, x_1), (y_{-1}, y_0, y_1)) \mapsto \bar{x}_{-1}y_1 + \bar{x}_0y_0 + \bar{x}_1y_{-1}$$

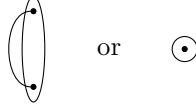
The advantage of taking this peculiar hermitian form is that the associated involution preserves the group of upper triangular matrices. As Lemma 2.3 shows, up to isomorphism, there is only one “type” of non-split, quasi-split algebraic group of relative rank 1 over local fields. Hence, choosing the above hermitian form is in fact not restrictive.

We can now describe quasi-split algebraic groups of relative rank 1 (recall that by the convention of this section, all our algebraic groups are absolutely simple, simply connected, algebraic groups over a local field).

**Lemma 2.3.** *Let  $K$  be a local field and let  $G$  be a quasi-split algebraic group of relative rank 1 over  $K$ . Then  $G$  is one of the following group:*

1.  $\mathrm{SL}_2$  over  $K$ .
2.  $\mathrm{SU}_3^{L/K}$ , where  $L$  is as in Definition 2.1.

*Proof.* If  $G$  is quasi-split, then by definition, its anisotropic kernel is trivial. Hence, by [Tit66, 2.7.1, Theorem 2],  $G$  is entirely determined (up to  $K$ -isomorphism) by its Dynkin diagram together with the  $*$ -action on it (or in other words,  $G$  is determined by its index). Also note that the number of orbit under this  $*$ -action is the relative rank, so that according to [Tit66, Table II], the only possibilities for the index are



The first index is the index of  $\mathrm{SU}_3^{L/K}$ , where  $L$  is any separable quadratic extension of  $K$ , while the second index is the index of  $\mathrm{SL}_2$ .  $\square$

## 2.2 The algebraic group $\mathrm{SL}_2(D)$

As outlined in the introduction, treating the case of the group  $\mathrm{SL}_2(D)$  (where  $D$  is a finite dimensional central division algebra) is very close to treating the case of  $\mathrm{SL}_2(K)$ , so that we decided to include this case as well. Let us recall the definition of the group  $\mathrm{SL}_2(D)$ .

**Definition 2.4.** Let  $D$  be a finite dimensional central division algebra over  $K$ . We define the group  $\mathrm{SL}_2(D) = \{u \in \mathrm{End}_D(D^2) \mid \mathrm{Nrd}(u) = 1\}$ , where  $\mathrm{Nrd}(u)$  stands for the reduced norm of  $u$  (we recall the definition of the reduced norm in Definition B.4), and  $D^2$  is considered as a right  $D$ -vector space.

Let us stress again that the case of main interest is the case of quasi-split groups, i.e. the case  $D = K$ . We advice the reader to consider only this case in a first reading.

When  $D = K$ , the group  $\mathrm{SL}_2(K)$  is the group of rational points of a closed subspace  $\mathrm{SL}_2$  of the affine space  $\mathbf{A}_K^4$  defined by the polynomial equation  $\det - 1$ . It is then straightforward to check that  $\mathrm{SL}_2$  is indeed an algebraic group over  $K$ .

For arbitrary  $D$ , it is well-known that  $\mathrm{SL}_2(D)$  can be seen as the group of rational point of an algebraic group over  $K$ . We recall in Appendix B the standard facts about division algebras, and we also discuss in Appendix C the representation of  $\mathrm{SL}_2(D)$  as an algebraic group over  $K$ .

## 3 The Bruhat–Tits tree of $\mathrm{SL}_2(D)$ and $\mathrm{SU}_3$

The aim of this section is to give a streamlined definition of the Bruhat–Tits tree associated with  $\mathrm{SL}_2(D)$  and  $\mathrm{SU}_3$ , together with the action on it. As outlined in the introduction, our definition of the Bruhat–Tits tree follows [BT72, §7].

In order to be as efficient as possible, we only describe concretely the objects needed, and give unmotivated definitions. Our description is easily obtained from the explicit description given in [BT72, §10], and we give in Appendix A more details about the connection with [BT72].

Recall from the introduction (or from general Bruhat–Tits theory) that the Bruhat–Tits tree  $\mathcal{I}$  should be isomorphic to  $G(K) \times \mathbf{R}/\sim$ . For  $x \in \mathbf{R}$ , we define a group  $P_x \leq G(K)$  which will eventually turn out to be the stabiliser of  $[(\mathrm{Id}, x)] \in \mathcal{I}$  (see Remark 3.11).

**Definition 3.1.** Let  $D$  be a finite dimensional central division algebra over  $K$  and let  $g$  be a  $n \times n$  matrix with coefficients in  $D$ . Given a  $n \times n$  matrix  $m$  with coefficient in  $\mathbf{R}$ , we say that  $g$  has a valuation greater than  $m$  if  $\omega(g_{ij}) \geq m_{ij}$  (for all  $i, j \in \{1, \dots, n\}$ ), and we denote it by  $\omega(g) \geq m$ .

**Definition 3.2.** In the  $\mathrm{SL}_2(D)$  case, for  $x \in \mathbf{R}$ , we define

$$P_x = \{g \in \mathrm{SL}_2(D) \mid \omega(g) \geq \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}\}$$

The definition of  $P_x$  in the  $\mathrm{SU}_3$  case is less straightforward when the residue characteristic is 2 and the extension  $L$  is ramified. Following [BT84a, 4.3.3], we define a parameter to handle the complication.



**Lemma 3.3.** *Let  $L$  be a separable quadratic extension of  $K$ . There exists  $t \in L$  and  $\alpha, \beta \in K$  such that:*

1.  $L = K[t]$  and  $t^2 - \alpha t + \beta = 0$ .
2.  $\omega(\beta) = 0$  when  $L$  is unramified, and  $\beta$  is a uniformiser of  $K$  when  $L$  is ramified.
3.  $\alpha = 0$ , or  $0 = \omega(\beta) = \omega(\alpha) < \omega(2)$ , or  $0 < \omega(\beta) \leq \omega(\alpha) \leq \omega(2)$ .

*Proof.* See [BT84a, Lemme 4.3.3, (ii)]. The fact that  $\alpha$  can be chosen so that  $\omega(\alpha) = 0$  in the unramified case is a direct consequence of the theory of unramified extensions of local fields (see for example [FV02, Chapter II, Section 3.2, Proposition]). With this in mind, the equivalence with [BT84a, Lemme 4.3.3, (ii)] is clear.  $\square$

**Remark 3.4.** To make Lemma 3.3 possibly clearer, let us state what is the valuation of  $\alpha$  on a case-by-case analysis:

1. If  $L$  is unramified,  $\begin{cases} \alpha = 0 \text{ if the residue characteristic is not } 2 \\ \omega(\alpha) = 0 \text{ if the residue characteristic is } 2 \end{cases}$
2. If  $L$  is ramified,  $\begin{cases} \alpha = 0 \text{ if the residue characteristic is not } 2 \\ \alpha = 0 \text{ or } 0 < \omega(\alpha) \leq \omega(2) \text{ if the residue characteristic is } 2 \end{cases}$

The only difference between Remark 3.4 and Lemma 3.3 is that the latter allows the possibility that  $\alpha = 0$  in the unramified residue characteristic 2 case. But this clearly cannot happen.

**Definition 3.5.** Let  $L$  be a separable quadratic extension of  $K$ , and let  $t, \alpha, \beta$  be as in Lemma 3.3. Let  $l = t\alpha^{-1} \in L$  if  $\alpha \neq 0$ , and  $l = \frac{1}{2} \in L$  if  $\alpha = 0$ , where  $\alpha$  is as in Lemma 3.3 (note that  $\alpha = 0$  implies  $2 \neq 0$  in  $K$ , since  $L$  is assumed to be a separable extension). We then define  $\gamma = -\frac{1}{2}\omega(l) \in \mathbf{R}$ .

**Remark 3.6.** Note that  $\gamma \geq 0$ . Furthermore, in view of Remark 3.4,  $\gamma > 0$  if and only if the residue characteristic is 2 and  $L$  is a ramified extension.

**Definition 3.7.** In the  $\mathrm{SU}_3^{L/K}$  case, let  $\gamma$  be the parameter associated with the extension  $L$  of  $K$  as in Definition 3.5. For  $x \in \mathbf{R}$ , we define

$$P_x = \{g \in \mathrm{SU}_3^{L/K}(K) \mid \omega(g) \geq \begin{pmatrix} 0 & -\frac{x}{2} - \gamma & -x \\ \frac{x}{2} + \gamma & 0 & -\frac{x}{2} + \gamma \\ x & \frac{x}{2} - \gamma & 0 \end{pmatrix}\}$$

The final ingredient in the definition of the Bruhat–Tits tree is the definition of a subgroup  $N$ , together with its affine action on  $\mathbf{R}$

**Definition 3.8.** 1. In the  $\mathrm{SL}_2(D)$  case, consider the following subsets

- (a)  $T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in D^\times \right\} < \mathrm{SL}_2(D)$
- (b)  $M = \left\{ \begin{pmatrix} 0 & -x \\ x^{-1} & 0 \end{pmatrix} \mid x \in D^\times \right\} \subset \mathrm{SL}_2(D)$

2. In the  $\mathrm{SU}_3^{L/K}$  case, consider the following subsets

- (a)  $T = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1}\bar{x} & 0 \\ 0 & 0 & \bar{x}^{-1} \end{pmatrix} \mid x \in L^\times \right\} < \mathrm{SU}_3^{L/K}(K)$
- (b)  $M = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & -x^{-1}\bar{x} & 0 \\ \bar{x}^{-1} & 0 & 0 \end{pmatrix} \mid x \in L^\times \right\} \subset \mathrm{SU}_3^{L/K}(K)$

In both cases, let  $N = T \sqcup M$ .



**Definition 3.9.** In both cases, we define a map  $\nu: N \rightarrow \text{Aff}(\mathbf{R})$  as follows. In the  $\text{SL}_2(D)$  case (respectively the  $\text{SU}_3$  case), for  $m = \begin{pmatrix} 0 & -x \\ x^{-1} & 0 \end{pmatrix} \in M$  (respectively  $m = \begin{pmatrix} 0 & 0 & x \\ 0 & -x^{-1} & 0 \\ \bar{x}^{-1} & 0 & 0 \end{pmatrix} \in M$ ),  $\nu(m)$  is the reflection through  $-\omega(x)$ , while for  $t = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in T$  (respectively for  $t = \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & \bar{x}^{-1} \end{pmatrix} \in T$ ),  $\nu(t)$  is the translation by  $-2\omega(x)$ .

We finally arrive at the definition of the Bruhat–Tits tree.

**Definition 3.10** ([BT72, 7.4.1 and 7.4.2]). Let  $G$  be either  $\text{SL}_2(D)$  or  $\text{SU}_3(K)$ . Define an equivalence relation on  $G \times \mathbf{R}$  as follows:  $(g, x) \sim (h, y)$  if and only if there exists  $n \in N$  such that  $y = \nu(n)(x)$  and  $g^{-1}hn \in P_x$ . The Bruhat–Tits tree of  $G$  is  $\mathcal{I} = G \times \mathbf{R} / \sim$ , and  $[(g, x)]$  stands for the equivalence class of  $(g, x)$  in  $\mathcal{I}$ . The group  $G$  acts on  $\mathcal{I}$  by multiplication on the first component.

**Remark 3.11.** We discuss in Appendix A why our groups  $P_x$  coincide with the groups  $\hat{P}_x$  appearing in the definition of the Bruhat–Tits building in [BT72, 7.4.1 and 7.4.2]. Since the definition of  $N$  together with its action  $\nu$  on  $\mathbf{R}$  also coincide with [BT72] (see also Appendix A for more details), the space  $\mathcal{I}$  of Definition 3.10 is really the Bruhat–Tits building of  $G$  as defined in [BT72]. In particular, for  $g \in G$ , the map  $f_g: \mathbf{R} \rightarrow \mathcal{I}: x \mapsto g \cdot [\text{Id}, x]$  is injective (by the discussion in [BT72], below Definition 7.4.2), an apartment of  $\mathcal{I}$  is a subset of the form  $f_g(\mathbf{R})$  for some  $g \in G$ , and we can endow  $\mathcal{I}$  with a metric which gives the usual metric on  $\mathbf{R}$  when restricted to any apartment. Furthermore, in view of [BT72, Proposition 7.4.4],  $P_x$  is in fact the stabiliser of  $[(\text{Id}, x)] \in \mathcal{I}$ .

**Remark 3.12.** The metric space  $\mathcal{I}$  is indeed a tree, whose regularity depends on  $G$ . If  $G = \text{SL}_2(D)$  (respectively  $\text{SU}_3^{L/K}(K)$  where  $L$  is ramified), then  $\mathcal{I}$  is the regular tree of degree  $|\overline{D}|+1$  (respectively  $|\overline{K}|+1$ ), while if  $G = \text{SU}_3^{L/K}(K)$  with  $L$  unramified, then  $\mathcal{I}$  is the semiregular tree of bidegree  $(|\overline{K}|^3+1; |\overline{K}|+1)$ . Indeed, this follows from the fact that our definition of  $\mathcal{I}$  agrees with the one given in [BT72, 7.4.1 and 7.4.2], and from the tables in [Tit79, 4.2 and 4.3].

**Remark 3.13.** Note that in Definition 3.10, it is equivalent to say that  $(g, x) \sim (h, y)$  if and only if for all  $\tilde{n} \in N$  such that  $\nu(\tilde{n})(x) = y$ , we have  $g^{-1}h\tilde{n} \in P_x$ . Indeed, if there exists  $n \in N$  such that  $\nu(n)(x) = y$  and  $g^{-1}hn \in P_x$ , let  $\tilde{n}$  be any element of  $N$  such that  $\nu(\tilde{n})(x) = y$ . Then  $g^{-1}h\tilde{n} = g^{-1}hnn^{-1}\tilde{n}$ . But  $n^{-1}\tilde{n}$  stabilises  $[(\text{Id}, x)]$ , and hence belongs to  $P_x$  by Remark 3.11. Thus,  $g^{-1}hnn^{-1}\tilde{n}$  belongs to  $P_x$  as well, as wanted.

We pass to another equivalent definition of the Bruhat–Tits tree, which can be thought of as a pointed version of  $\mathcal{I}$  around  $[(\text{Id}, 0)]$ .

**Definition 3.14.** In the  $\text{SL}_2(D)$  case or the  $\text{SU}_3(K)$  case, we define an equivalence relation on  $P_0 \times \mathbf{R}$  as follows:  $(g, x) \sim_0 (h, y)$  if and only if there exists  $n \in N \cap P_0$  such that  $y = \nu(n)(x)$  and  $g^{-1}hn \in P_x \cap P_0$ . The Bruhat–Tits tree of  $G$  centred at 0 is  $\mathcal{I}_0 = P_0 \times \mathbf{R} / \sim_0$ , and  $[(g, x)]_0$  stands for the equivalence class of  $(g, x)$  in  $\mathcal{I}_0$ . The group  $P_0$  acts on  $\mathcal{I}_0$  by multiplication on the first component.

To prove that  $\mathcal{I}_0$  is naturally in equivariant bijection with  $\mathcal{I}$ , we need the following observation.

**Lemma 3.15.** Let  $g, h \in P_0$ , and let  $x, y \in \mathbf{R}$ . If  $(g, x) \sim (h, y)$ , there exists  $n \in N \cap P_0$  such that  $\nu(n)(x) = y$ .

*Proof.* Recall that  $P_0$  is the stabiliser of  $[(\text{Id}, 0)] \in \mathcal{I}$  in  $G$  (see Remark 3.11). Since  $G$  acts by isometries on  $\mathcal{I}$ , and since  $g, h \in P_0$ , we have

$$\begin{aligned} |x| &= d_{\mathcal{I}}([(g, x)], [(\text{Id}, 0)]) = d_{\mathcal{I}}([(g, x)], [(\text{Id}, 0)]) \\ |y| &= d_{\mathcal{I}}([(h, y)], [(\text{Id}, 0)]) = d_{\mathcal{I}}([(h, y)], [(\text{Id}, 0)]) \end{aligned}$$

where  $d_{\mathcal{I}}$  denotes the distance in the metric space  $\mathcal{I}$  (see Remark 3.11). But if  $(g, x) \sim (h, y)$ , we have in particular  $d_{\mathcal{I}}([(g, x)], [(\text{Id}, 0)]) = d_{\mathcal{I}}([(h, y)], [(\text{Id}, 0)])$ , and hence  $|x| = |y|$ . Thus, the existence of  $n \in N \cap P_0$  such that  $\nu(n)(x) = y$  follows from Definition 3.9.  $\square$

**Lemma 3.16.** *Let  $G$  be either  $\mathrm{SL}_2(D)$  or  $\mathrm{SU}_3(K)$ . The map  $\mathcal{I}_0 \rightarrow \mathcal{I}: [(g, x)]_0 \mapsto [(g, x)]$  is a  $(P_0 \hookrightarrow G)$ -equivariant bijection.*

*Proof.* • Injectivity: assume  $(g, x) \sim (h, y)$ , where  $g, h$  are in  $P_0$ . By Lemma 3.15, there exists  $n \in N \cap P_0$  such that  $y = \nu(n)(x)$  and since  $(g, x) \sim (h, y)$ ,  $g^{-1}hn \in P_x$  by Remark 3.13. But  $g^{-1}hn$  also belongs to  $P_0$ , so that  $(g, x) \sim_0 (h, y)$ , as wanted.

• Surjectivity: let  $[(g, x)] \in \mathcal{I}$ . Since  $G$  acts strongly transitively on  $\mathcal{I}$  ([BT72, Corollaire 7.4.9]), there exists  $h \in P_0$  such that  $h \cdot [(g, x)] = [(\mathrm{Id}, y)]$ , for some  $y \in \mathbf{R}$ . Hence,  $[(g, x)]$  is the image of  $[(h^{-1}, y)]_0 \in \mathcal{I}_0$ .

• Equivariance: the image of  $h \cdot [(g, x)]_0$  is  $[(hg, x)] = h \cdot [(g, x)]$ .  $\square$

## 4 Local description of the Bruhat–Tits tree

We now aim to give a local description of balls of the Bruhat–Tits tree, together with the group action on it. Recall that the ball of radius 1 around  $[(\mathrm{Id}, 0)] \in \mathcal{I}$  (together with the action of  $P_0$  on it), is in some sense encoded in  $P_0$  considered over the residue field, i.e. over  $\mathcal{O}_K/\mathfrak{m}_K$  (see [BT84a, Théorème 4.6.33] for a precise meaning). It is then natural to think that more generally, the ball of radius  $r$  around  $[(\mathrm{Id}, 0)] \in \mathcal{I}$  (together with the action of  $P_0$  on it) is encoded in  $P_0$  considered over the ring  $\mathcal{O}_K/\mathfrak{m}_K^r$ . We prove in this section that this is indeed true.

### 4.1 Local models for the Bruhat–Tits tree

We just mimic the definition of the Bruhat–Tits tree, except that the coefficients of all groups under consideration are now taken in the ring  $\mathcal{O}_D/\mathfrak{m}_D^r$  (or  $\mathcal{O}_L/\mathfrak{m}_L^r$  in the  $\mathrm{SU}_3$  case). All groups defined in this section are adorned by the superscript  $0, r$  to reflect the fact that they are local version around 0 of radius  $r$ .

**Definition 4.1.** In the  $\mathrm{SL}_2(D)$  case, let  $r \in \mathbf{N} \cup \{\infty\}$ . We only need to describe balls of radius  $rd$ , where  $d$  is the degree of  $D$  over its center  $K$ . Let  $x \in [-\omega(\pi_D^{rd}), \omega(\pi_D^{rd})]$ . Note that the valuation  $\omega$  induces a valuation on  $\mathcal{O}_D/\mathfrak{m}_D^{rd}$ , that we still denote  $\omega$ . By convention,  $\mathfrak{m}_D^\infty = (0)$ . Mimicking Definition 3.2, we define  $P_x^{0, rd} = \{g \in \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) \mid \omega(g) \geq \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}\}$  (see Definition C.2 for the definition of  $\mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd})$ ). When  $D = K$ , we obtain the group  $\mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r)$  in its usual meaning, i.e. the group of  $2 \times 2$  matrices with coefficient in  $\mathcal{O}_K/\mathfrak{m}_K^r$  having determinant 1).

We also need the local version of the subgroup  $N$ .

**Definition 4.2.** In the  $\mathrm{SL}_2(D)$  case, we define

1.  $H^{0, rd} = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^r) \mid \omega(x) = 0 \right\}$
2.  $M^{0, rd} = \left\{ \begin{pmatrix} 0 & -x \\ x^{-1} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^r) \mid \omega(x) = 0 \right\}$

And then, we set  $N^{0, rd} = H^{0, rd} \sqcup M^{0, rd}$

In the  $\mathrm{SU}_3$  case, some complications arise due to the fact that the group  $P_0$  of Definition 3.7 is not naturally described as living in  $\mathrm{SL}_3(\mathcal{O}_L)$  when the parameter  $\gamma$  of Definition 3.5 is strictly positive, i.e. when the residue characteristic is 2 and the extension  $L$  is ramified. This is related to the fact that if one considers the algebraic group  $\underline{G} = \underline{\mathrm{SU}}_3^{L/K}$  over  $\mathcal{O}_K$  as in Definition 4.12 (which is possible since the equations  $\det(g) - 1$  and  ${}^S\bar{g}g - \mathrm{Id}$  only involve coefficients belonging to  $\mathcal{O}_K$ ), then it is not smooth (as an  $\mathcal{O}_K$ -scheme) if and only if the residue characteristic is 2 and the extension  $L$  is ramified. Indeed, in this case,  $\dim_{\overline{K}} \mathrm{Lie}(\underline{G}_{\overline{K}}) = \dim_K \mathrm{Lie}(\underline{G}_K) + 3$ , while smoothness of  $\underline{\mathrm{SU}}_3^{L/K}$  when  $\gamma = 0$  is proved in Theorem 4.13.

By contrast, the correct definition of the local version of the Bruhat–Tits tree in the  $\mathrm{SU}_3$  case when  $\gamma = 0$  is the “natural” one.

**Definition 4.3.** In the  $\text{SU}_3$  case when  $\gamma = 0$ , let  $r \in \mathbf{N} \cup \{\infty\}$ . Let  $x \in [-\omega(\pi_L^r), \omega(\pi_L^r)]$ . Note that the valuation  $\omega$  induces a valuation on  $\mathcal{O}_L/\mathfrak{m}_L^r$ , that we still denote  $\omega$ . Also, the Galois action on  $L$  induce an action on  $\mathcal{O}_L/\mathfrak{m}_L^r$ , that we also denote by  $x \mapsto \bar{x}$ . By convention,  $\mathfrak{m}_L^\infty = (0)$ . Mimicking Definition 3.7, we define  $P_x^{0,r} = \{g \in \text{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^r) \mid {}^S \bar{g}g = \text{Id}, \omega(g) \geq \begin{pmatrix} 0 & -\frac{x}{2} & -x \\ \frac{x}{2} & 0 & -\frac{x}{2} \\ x & \frac{x}{2} & 0 \end{pmatrix}\}$ .

Again, we need the local version of the subgroup  $N$ .

**Definition 4.4.** 1.  $H^{0,r} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1}\bar{x} & 0 \\ 0 & 0 & \bar{x}^{-1} \end{pmatrix} \in \text{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(x) = 0 \right\}$

2.  $M^{0,r} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & -x^{-1}\bar{x} & 0 \\ \bar{x}^{-1} & 0 & 0 \end{pmatrix} \in \text{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(x) = 0 \right\}$

And then, we set  $N^{0,r} = H^{0,r} \sqcup M^{0,r}$ .

When  $\gamma > 0$  (i.e. when the residue characteristic is 2 and  $L$  is ramified), we only need to give the local description for small radii. We introduce a new parameter which controls the meaning of small in this case.

**Definition 4.5.** Set  $i_0 = \min\{r \in \mathbf{N} \mid \omega(\pi_L^r) \geq \gamma\}$ . Equivalently, let  $\alpha$  be as in Lemma 3.3. If  $\alpha = 0$  (respectively if  $\alpha \neq 0$ ),  $i_0$  is such that  $\omega(\pi_K^{i_0}) = \omega(2)$  (respectively  $\omega(\pi_K^{i_0}) = \omega(\alpha)$ ).

**Definition 4.6.** In the  $\text{SU}_3$  case when  $\gamma > 0$ , let  $r \in \mathbf{N}$  be such that  $r \leq 2i_0$ . Let  $x \in [-\omega(\pi_L^r), \omega(\pi_L^r)]$ . Note that the valuation  $\omega$  induces a valuation on  $\mathcal{O}_L/\mathfrak{m}_L^r$ , that we still denote  $\omega$ . we define  $P_x^{0,r} = \{g \in \text{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(g) \geq \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}\}$

We also need the local version of the subgroup  $N$ .

**Definition 4.7.** In the  $\text{SU}_3$  case when  $\gamma > 0$  and for  $r \leq 2i_0$ , we define

1.  $H^{0,r} = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in \text{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(x) = 0 \right\}$

2.  $M^{0,r} = \left\{ \begin{pmatrix} 0 & -x \\ x^{-1} & 0 \end{pmatrix} \in \text{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(x) = 0 \right\}$

And then, we set  $N^{0,r} = H^{0,r} \sqcup M^{0,r}$

We can also easily define an action of  $N^{0,r}$  by affine isometries on  $\mathbf{R}$ .

**Definition 4.8.** In all cases ( $\text{SL}_2(D)$  and  $\text{SU}_3$  for  $\gamma \geq 0$ ), we let  $H^{0,r}$  acts trivially on  $\mathbf{R}$ , and we let all elements of  $M^{0,r}$  act as a reflection through  $0 \in \mathbf{R}$ . This gives an affine action of  $N^{0,r}$  on  $\mathbf{R}$ , and we denote again the resulting map  $N^{0,r} \rightarrow \text{Aff}(\mathbf{R})$  by  $\nu$ .

We are now able to give a definition of the ball of radius  $r$  around  $[(\text{Id}, 0)] \in \mathcal{I}$  which only depends on the ring  $\mathcal{O}/\mathfrak{m}^r$ , and not on the whole division algebra  $D$  or the field  $L$ .

**Definition 4.9.** Let  $r \in \mathbf{N} \cup \{\infty\}$ . In the  $\text{SL}_2(D)$  case (respectively the  $\text{SU}_3$  case), let  $\pi = \pi_D$  and  $d = \sqrt{[D : K]}$  (respectively  $\pi = \pi_L$  and  $d = 1$ ). Also assume that  $r \leq 2i_0$  in the  $\text{SU}_3$  case when  $\gamma > 0$ . We define an  $rd$ -local equivalence on  $P_0^{0,rd} \times [-\omega(\pi^{rd}), \omega(\pi^{rd})]$  as follows. For  $g, h \in P_0^{0,rd}$  and  $x, y \in [-\omega(\pi^{rd}), \omega(\pi^{rd})]$

$$(g, x) \sim_{0,rd} (h, y) \Leftrightarrow \text{there exists } n \in N^{0,rd} \text{ such that } \nu(n)(x) = y \text{ and } g^{-1}hn \in P_x^{0,rd}$$

The resulting space  $\mathcal{I}^{0,rd} = P_0^{0,rd} \times [-\omega(\pi^{rd}), \omega(\pi^{rd})] / \sim_{0,rd}$  is called the local Bruhat–Tits tree of radius  $rd$  around 0, and  $[(g, x)]^{0,rd}$  stands for the equivalence class of  $(g, x)$  in  $\mathcal{I}^{0,rd}$ . The group  $P_0^{0,rd}$  acts on  $\mathcal{I}^{0,rd}$  by multiplication on the first component.

**Remark 4.10.** Note that as for Definition 3.10, it is equivalent to say that  $(g, x) \sim_{0,rd} (h, y)$  if and only if for all  $\tilde{n} \in N^{0,rd}$  such that  $\nu(\tilde{n})(x) = y$ , we have  $g^{-1}h\tilde{n} \in P_x^{0,rd}$ . Indeed, if there exists  $n \in N^{0,rd}$  such that  $\nu(n)(x) = y$  and  $g^{-1}hn \in P_x^{0,rd}$ , let  $\tilde{n}$  be any element of  $N^{0,rd}$  such that  $\nu(\tilde{n})(x) = y$ . We have  $g^{-1}h\tilde{n} = g^{-1}hnn^{-1}\tilde{n}$ , and a case-by-case analysis shows that  $n^{-1}\tilde{n} \in P_x^{0,rd}$ . Hence  $g^{-1}hnn^{-1}\tilde{n}$  belongs to  $P_x^{0,rd}$  as well, as wanted.

## 4.2 Integral models

We have just defined the space  $\mathcal{I}^{0,rd}$ , where  $d$  is the degree of  $D$  in the  $\mathrm{SL}_2(D)$  case, and is equal to one otherwise. In order to show that it encodes the ball of radius  $rd$  together with the action of  $P_0$  on it (as will be done in Theorem 4.32), we need to prove that there exists a surjective homomorphism  $P_0 \rightarrow P_0^{0,rd}$ . In the  $\mathrm{SL}_2(D)$  case (respectively the  $\mathrm{SU}_3$  case when  $\gamma = 0$ ), the homomorphism  $P_0 \rightarrow P_0^{0,rd}$  is just the one induced by the projection  $\mathcal{O}_D \rightarrow \mathcal{O}_D/\mathfrak{m}_D^{rd}$  (respectively  $\mathcal{O}_L \rightarrow \mathcal{O}_L/\mathfrak{m}_L^r$ ). But in the  $\mathrm{SU}_3$  case when  $\gamma > 0$ , even the existence of such a homomorphism is not obvious at first sight.

We solve the question by defining (for each case separately) a smooth  $\mathcal{O}_K$ -scheme  $\underline{G}$ , such that  $\underline{G}(\mathcal{O}_K) \cong P_0$  and  $\underline{G}(\mathcal{O}_K/\mathfrak{m}_K^r) \cong P_0^{0,rd}$  (where  $\epsilon = 2$  in the  $\mathrm{SU}_3$  case when  $L$  is ramified, and is equal to 1 otherwise). Then the desired surjectivity follows by an application of Hensel's lemma for smooth schemes (that we recall in Theorem 4.25).

The smooth  $\mathcal{O}_K$ -scheme  $\underline{G}$  is in fact the Bruhat–Tits integral model  $\hat{\mathfrak{B}}_\varphi$  associated with a standard valuation  $\varphi$  (see [BT84a, 4.6.26]). A potential interest of this section is that we also give an explicit description of this integral model in the more complicated case of  $\mathrm{SU}_3$  when  $\gamma > 0$ . But let us begin with the  $\mathrm{SL}_2(D)$  case and the  $\mathrm{SU}_3$  case when  $\gamma = 0$ .

**Definition 4.11.** Let  $\underline{\mathrm{SL}}_2$  be the group  $\mathrm{SL}_2$  considered over  $\mathcal{O}_K$ . Concretely, this is the  $\mathcal{O}_K$ -scheme associated with the  $\mathcal{O}_K$ -algebra  $\mathcal{O}_K[\underline{\mathrm{SL}}_2] = \mathcal{O}_K[X_{11}, X_{12}, X_{21}, X_{22}]/(X_{11}X_{22} - X_{12}X_{21} - 1)$ . In the case of a central division algebra of degree  $d > 1$  over  $K$ , the definition of an integral model  $\underline{\mathrm{SL}}_{2,D}$  over  $\mathcal{O}_K$  is a bit less straightforward to define. We give it in the appendix (see Definition C.3).

**Definition 4.12.** When the parameter  $\gamma$  associated with  $L/K$  is 0, let  $\underline{\mathrm{SU}}_3^{L/K}$  be the group  $\mathrm{SU}_3$  considered over  $\mathcal{O}_K$ . We often omit the superscript  $L/K$ . Concretely,  $\underline{\mathrm{SU}}_3$  is the  $\mathcal{O}_K$ -scheme associated with the  $\mathcal{O}_K$ -algebra  $\mathcal{O}_K[\underline{\mathrm{SU}}_3] = \mathcal{O}_K[X_{ij}^{kl}]/I$  ( $i, j \in \{1, 2, 3\}$ ,  $k, l \in \{1, 2\}$ ), where  $I$  is the ideal generated by the following equations

$$\begin{aligned} \text{For all } i, j \in \{1, 2, 3\}, \quad & \begin{cases} X_{ij}^{12} = -\beta X_{ij}^{21} \\ X_{ij}^{22} = X_{ij}^{11} + \alpha X_{ij}^{21} \end{cases} \\ & \sum_{\sigma \in \mathrm{Sym}(3)} [(-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^3 X_{i\sigma(i)}] - 1 \\ & \begin{pmatrix} \overline{X}_{33} & \overline{X}_{23} & \overline{X}_{13} \\ \overline{X}_{32} & \overline{X}_{22} & \overline{X}_{12} \\ \overline{X}_{31} & \overline{X}_{21} & \overline{X}_{11} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Here  $\alpha$  and  $\beta$  are as in Lemma 3.3, so that the first equations encode the ring embedding  $\mathcal{O}_L \hookrightarrow M_2(\mathcal{O}_K)$ . Also, for a  $2 \times 2$  matrix  $M = \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix}$ , we denote  $\overline{M} = \begin{pmatrix} M^{22} & -M^{12} \\ -M^{21} & M^{11} \end{pmatrix}$  (this operation reflects the conjugation on  $\mathcal{O}_L$ ). Finally note that a 1 (respectively a 0) in the above equations denotes the  $2 \times 2$  identity matrix (respectively the  $2 \times 2$  zero matrix), i.e. it corresponds to the  $1 \in L$  (respectively  $0 \in L$ ).

**Theorem 4.13.**  $\underline{\mathrm{SL}}_{2,D}$  and  $\underline{\mathrm{SU}}_3^{L/K}$  (when  $\gamma = 0$ ) are smooth  $\mathcal{O}_K$ -scheme.

*Proof.* Smoothness of  $\underline{\mathrm{SL}}_2$  over  $\mathcal{O}_K$  (and in fact of  $\underline{\mathrm{SL}}_n$  over any ring) is easily checked using the infinitesimal lifting criterion (see [TS16, Tag 02H6]). The case of  $\underline{\mathrm{SL}}_{2,D}$  is discussed in the appendix (see Theorem C.4).

We now prove the smoothness of  $\underline{\mathrm{SU}}_3$ . It suffices to prove that it is flat and that the fibres are smooth. The generic fibre is  $\mathrm{SU}_3^{L/K}$ , and is a form of  $\mathrm{SL}_3$ , hence is smooth over  $K$ . The closed fibre is the  $\overline{K}$ -functor  $(\underline{\mathrm{SU}}_3)_{\overline{K}}$  which associates to any  $\overline{K}$ -algebra  $R$  the group

$$(\underline{\mathrm{SU}}_3)_{\overline{K}}(R) = \{g \in \mathrm{SL}_3(R \otimes_{\overline{K}} \mathcal{O}_L/\mathfrak{m}_L^\epsilon) \mid {}^S \bar{g}g = \mathrm{Id}\}$$

where  $\epsilon = 1$  if  $L$  is unramified, and  $\epsilon = 2$  if  $L$  is ramified. When  $L$  is unramified, this algebraic group becomes isomorphic to  $\mathrm{SL}_3$  after base change to  $\overline{L}$ , and hence is smooth and connected. We now treat the ramified case. Let  $\mathrm{SO}_3$  be the special orthogonal group associated with the quadratic form  $(x_{-1}, x_0, x_1) \mapsto x_{-1}x_1 + x_0^2$ , considered over  $\overline{K}$ . More explicitly, for a  $\overline{K}$ -algebra  $R$ ,

$$(\mathrm{SO}_3)_{\overline{K}}(R) = \{g \in \mathrm{SL}_3(R) \mid {}^S g g = \mathrm{Id}\}$$

Since by assumption  $\gamma \neq 0$ , the characteristic of  $\overline{K}$  is not 2, and it is then well known that  $\mathrm{SO}_3$  is isomorphic to  $\mathrm{PGL}_2$  over  $\overline{K}$ , hence is smooth and connected of dimension 3. There exists a homomorphism of algebraic groups  $f: (\underline{\mathrm{SU}}_3)_{\overline{K}} \rightarrow (\mathrm{SO}_3)_{\overline{K}}$  induced by the homomorphism of  $\overline{K}$ -algebra  $\mathcal{O}_L/\mathfrak{m}_L^2 \rightarrow \overline{K}$ . The kernel of this map can be computed by hand, and we obtain that for any  $\overline{K}$ -algebra  $R$ ,

$$\ker f(R) = \{g \in \mathrm{SL}_3(R \otimes_{\overline{K}} \mathcal{O}_L/\mathfrak{m}_L^2) \mid g = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ g_{11}^{21} & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ g_{12}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ g_{13}^{21} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ g_{21}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2g_{11}^{21} & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ g_{12}^{21} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ g_{31}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ g_{21}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ g_{11}^{21} & 1 \end{pmatrix} \end{pmatrix}\}$$

This description makes it clear that  $\ker f$  is of dimension 5 and connected. Hence, using [DG70, II, §5, Proposition 5.1] (note that it does not use smoothness), we conclude that  $\dim(\underline{\mathrm{SU}}_3)_{\overline{K}} = 8$ . But we can also easily compute that the Lie algebra of  $(\underline{\mathrm{SU}}_3)_{\overline{K}}$  is

$$(\mathfrak{su}_3)_{\overline{K}} = \{g \in M_3(\mathcal{O}_L/\mathfrak{m}_L^2) \mid {}^S \bar{g} + g = 0, \mathrm{trace}(g) = 0\}$$

This is readily seen to be of dimension 8 (recall that we are in the case  $\gamma = 0$  and  $L$  ramified, so that the residue characteristic is not 2), and hence, we conclude that  $(\underline{\mathrm{SU}}_3)_{\overline{K}}$  is smooth, as wanted. Also note that the homomorphism  $f: (\underline{\mathrm{SU}}_3)_{\overline{K}} \rightarrow (\mathrm{SO}_3)_{\overline{K}}$  is surjective onto a connected algebraic group, with connected kernel, hence  $(\underline{\mathrm{SU}}_3)_{\overline{K}}$  is also connected.

It remains to prove flatness. Since  $\mathcal{O}_K$  is a prufferian ring, flatness is equivalent to being without torsion (see [BT84a, 2.2.2]). In other words, to prove flatness, it suffices to prove that  $(\underline{\mathrm{SU}}_3)_K$  is dense in  $(\underline{\mathrm{SU}}_3)_{\overline{K}}$ . Since we proved that  $(\underline{\mathrm{SU}}_3)_{\overline{K}}$  is connected, one can argue as in the conclusion of the proof of Lemma 4.22, when we show that  $\mathfrak{Y}_K$  is dense in  $\mathfrak{Y}$ .  $\square$

**Remark 4.14.** In passing, note that the group  $(\underline{\mathrm{SU}}_3^{L/K})_{\overline{K}}$  is not a reductive group over  $\overline{K}$  when  $L$  is ramified (as predicted by [BT84a, 4.6.31]). In fact, we just showed in the above proof that its reductive quotient is naturally described as the orthogonal group in 3 variables. Again, this might be seen as a reason for the complication of the ramified, residue characteristic 2, since philosophically, it involves orthogonal group in characteristic 2.

**Remark 4.15.** There is also a more direct way to prove the smoothness of  $\underline{\mathrm{SU}}_3$  in the unramified case, since in this case  $(\underline{\mathrm{SU}}_3)_{\mathcal{O}_L}$  is isomorphic to  $\mathrm{SL}_3$  over  $\mathcal{O}_L$ . But this does not work in the ramified case. Indeed, if  $(\underline{\mathrm{SU}}_3)_{\mathcal{O}_L}$  were isomorphic to  $\mathrm{SL}_3$  over  $\mathcal{O}_L$  in the ramified case, then its closed fibre  $(\underline{\mathrm{SU}}_3)_{\overline{K}}$  would be isomorphic to  $\mathrm{SL}_3$  over  $\overline{K} \cong \mathcal{O}_L/\mathfrak{m}_L$ , which is not true, as we have just seen in the above proof.

We now give the explicit equation of the integral model in the  $\mathrm{SU}_3$  case when  $\gamma > 0$ .

**Definition 4.16.** Let  $K[\mathrm{SU}_3^{L/K}]$  be the standard representation of the coordinate ring of  $\mathrm{SU}_3^{L/K}$ . More explicitly,  $K[\mathrm{SU}_3^{L/K}] = K[X_{ij}^{kl}]/I$  ( $i, j \in \{1, 2, 3\}, k, l \in \{1, 2\}$ ), where  $I$  is the ideal generated by the equations displayed in Definition 4.12. We also use the ring  $\mathcal{O}_K[\mathbf{A}^{36}] = \mathcal{O}_K[X_{ij}^{kl}]$  ( $i, j \in \{1, 2, 3\}, k, l \in \{1, 2\}$ ).

**Notation 4.17.** We use the following notations:  $\lambda_k = \begin{pmatrix} \pi_K^{k+1} & 0 \\ 0 & \pi_K^k \end{pmatrix}$ ,  $v_k = \begin{pmatrix} \pi_K^k & 0 \\ 0 & \pi_K^{k+1} \end{pmatrix}$  and  $\tau_k = \begin{pmatrix} \pi_K^k & 0 \\ 0 & \pi_K^k \end{pmatrix}$ .

Recall the definition of  $i_0$  in Definition 4.5, and let  $n_0 = \lfloor \frac{i_0}{2} \rfloor$ . The integral model depends on the parity of  $i_0$ . If  $i_0$  is odd, we define the  $\mathcal{O}_K$ -algebra map

$$\varphi_{i_0}: \mathcal{O}_K[\mathbf{A}^{36}] \rightarrow K[\mathrm{SU}_3^{L/K}]: \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \mapsto \begin{pmatrix} X_{11} & X_{12}\lambda_{n_0} & X_{13} \\ \lambda_{n_0}^{-1}X_{21} & \lambda_{n_0}^{-1}X_{22}\lambda_{n_0} & \lambda_{n_0}^{-1}X_{23} \\ X_{31} & X_{32}\lambda_{n_0} & X_{33} \end{pmatrix}$$

while if  $i_0$  is even, we define the  $\mathcal{O}_K$ -algebra map

$$\varphi_{i_0}: \mathcal{O}_K[\mathbf{A}^{36}] \rightarrow K[\mathrm{SU}_3^{L/K}]: \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \mapsto \begin{pmatrix} X_{11} & X_{12}\tau_{n_0} & X_{13} \\ \tau_{n_0}^{-1}X_{21} & X_{22} & \tau_{n_0}^{-1}X_{23} \\ X_{31} & X_{32}\tau_{n_0} & X_{33} \end{pmatrix}$$

**Remark 4.18.** The above notation for the map  $\varphi_{i_0}$  means that (for example in the  $i_0$  odd case)  $\varphi_{i_0}(Y_{11}) = X_{11}$ ,  $\varphi_{i_0}(Y_{21}) = \lambda_{n_0}^{-1}X_{21}$ , and so on.

**Definition 4.19.** Let  $\underline{\mathrm{SU}}_3^{L/K}$  be the closed subscheme of  $\mathbf{A}^{36}$  (over  $\mathcal{O}_K$ ) defined by the ideal  $\ker \varphi_{i_0}$ . We often omit the superscript  $L/K$  when it is not necessary to insist on the pair of field  $(K, L)$  under consideration.

**Remark 4.20.** Note that  $\varphi_{i_0}$  is just the equation for a base change. Also note that by definition,  $\underline{\mathrm{SU}}_3$  is the schematic adherence of  $\mathrm{SU}_3$  in  $\mathbf{A}^{36}$  (see [BT84a, 1.2.6] for the definition of the schematic adherence). Actually,  $\underline{\mathrm{SU}}_3$  is the integral model  $\hat{\mathfrak{G}}_\varphi$  associated in the sense of [BT84a, 4.6.26] to the standard valuation of  $\mathrm{SU}_3$ . The concrete description given here was found following the concrete description given in [BT87], see especially section 3.9 and the Theorem in section 5 in loc. cit. But we provide a concrete proof that  $\underline{\mathrm{SU}}_3$  is a smooth  $\mathcal{O}_K$ -group scheme, without referring to [BT87].

To not lengthen too much the paper, we now make all arguments when  $i_0$  is odd, the case  $i_0$  even being similar, if not simpler. A first important observation is that  $P_0 \cong \underline{\mathrm{SU}}_3(\mathcal{O}_K)$ .

**Lemma 4.21.** *The map  $\varphi_{i_0}$  gives an isomorphism  $\mathrm{SU}_3 \rightarrow (\underline{\mathrm{SU}}_3)_K$ , and the inverse image of  $(\underline{\mathrm{SU}}_3)_K(\mathcal{O}_K) \subset \{g \in \mathbf{A}^{36} \mid \omega(g_{ij}^{kl}) \geq 0\}$  is just  $\{g \in \mathrm{SU}_3(K) \mid \omega(g) \geq \begin{pmatrix} 0 & -\gamma & 0 \\ \gamma & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix}\}$ . In view of Definition 3.7, we indeed have that  $P_0 \cong \underline{\mathrm{SU}}_3(\mathcal{O}_K)$ .*

*Proof.* By definition, for  $g \in \mathrm{SU}_3(K)$ ,  $\varphi_{i_0}(g) = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \lambda_{n_0}^{-1} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \lambda_{n_0} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$ . For example, let us examine what we get for  $g_{12}$ . The  $2 \times 2$  matrix  $\begin{pmatrix} g_{12}^{11} & -\beta g_{12}^{12} \\ g_{12}^{21} & g_{12}^{11} + \alpha g_{12}^{21} \end{pmatrix}$  is thus sent to  $\begin{pmatrix} \pi_K^{n_0+1} g_{12}^{11} & -\pi_K^{n_0} \beta g_{12}^{12} \\ \pi_K^{n_0+1} g_{12}^{21} & \pi_K^{n_0} (g_{12}^{11} + \alpha g_{12}^{21}) \end{pmatrix}$ . All coefficients of this latter matrix are integral if and only if  $g_{12}^{11} \in (\pi_K^{-n_0})$  and  $g_{12}^{21} \in (\pi_K^{-n_0-1})$ . We have  $(\pi_K^{-n_0}) = (\pi_L^{-2n_0}) = (\pi_L^{-(i_0-1)})$  and  $(\pi_K^{-n_0-1}) = (\pi_L^{-2n_0-2}) = (\pi_L^{-(i_0+1)})$  (recall that we are just treating the case  $i_0$  odd). But by Definition 4.5,  $i_0$  is the smallest integer such that  $\omega(\pi_L^{i_0}) \geq \gamma$ . Hence, All coefficients of  $\begin{pmatrix} \pi_K^{n_0+1} g_{12}^{11} & -\pi_K^{n_0} \beta g_{12}^{12} \\ \pi_K^{n_0+1} g_{12}^{21} & \pi_K^{n_0} (g_{12}^{11} + \alpha g_{12}^{21}) \end{pmatrix}$  are integral if and only if  $\omega(g_{12}) \geq -\gamma$ . The other cases are similar.  $\square$

**Lemma 4.22.** *The ideal defining  $\underline{\mathrm{SU}}_3$  in  $\mathbf{A}^{36}$  is generated by the following equations*

1. *If  $i_0$  is odd*

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \begin{pmatrix} \tau_{i_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau_{i_0} \end{pmatrix} \begin{pmatrix} \bar{Y}_{33} & \bar{Y}_{23} & \bar{Y}_{13} \\ \bar{Y}_{32} & \bar{Y}_{22} & \bar{Y}_{12} \\ \bar{Y}_{31} & \bar{Y}_{21} & \bar{Y}_{11} \end{pmatrix} = \begin{pmatrix} \tau_{i_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau_{i_0} \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} \bar{Y}_{33} & \bar{Y}_{23} & \bar{Y}_{13} \\ \bar{Y}_{32} & \bar{Y}_{22} & \bar{Y}_{12} \\ \bar{Y}_{31} & \bar{Y}_{21} & \bar{Y}_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{i_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{i_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2)$$

$$\frac{2}{\pi_K^{i_0}} (Y_{31}^{11} Y_{11}^{11} + \beta Y_{31}^{21} Y_{11}^{21}) + \frac{\alpha}{\pi_K^{i_0}} (Y_{31}^{21} Y_{11}^{11} + Y_{11}^{21} Y_{31}^{11}) = -(\bar{Y}_{21} Y_{21})^{11} \quad (3)$$

$$\frac{2}{\pi_K^{i_0}}(Y_{13}^{11}Y_{33}^{11} + \beta Y_{13}^{21}Y_{33}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(Y_{13}^{21}Y_{33}^{11} + Y_{33}^{21}Y_{13}^{11}) = -(\bar{Y}_{23}Y_{23})^{11} \quad (4)$$

$$\text{for } (i, j) \in \{(1, 1); (1, 3); (3, 1); (3, 3)\}, \begin{cases} Y_{ij}^{12} = -\beta Y_{ij}^{21} \\ Y_{ij}^{22} = Y_{ij}^{11} + \alpha Y_{ij}^{21} \end{cases} \quad (5)$$

$$\begin{cases} Y_{21}^{12} = -\frac{\beta}{\pi_K} Y_{21}^{21} \\ Y_{21}^{22} = \pi_K Y_{21}^{11} + \alpha Y_{21}^{21} \end{cases} \quad \begin{cases} Y_{23}^{12} = -\frac{\beta}{\pi_K} Y_{23}^{21} \\ Y_{23}^{22} = \pi_K Y_{23}^{11} + \alpha Y_{23}^{21} \end{cases} \quad (6)$$

$$\begin{cases} Y_{12}^{12} = -\frac{\beta}{\pi_K} Y_{12}^{21} \\ \pi_K Y_{12}^{22} = Y_{12}^{11} + \alpha Y_{12}^{21} \end{cases} \quad \begin{cases} Y_{32}^{12} = -\frac{\beta}{\pi_K} Y_{32}^{21} \\ \pi_K Y_{32}^{22} = Y_{32}^{11} + \alpha Y_{32}^{21} \end{cases} \quad (7)$$

$$\begin{cases} \pi_K Y_{22}^{12} = -\frac{\beta}{\pi_K} Y_{22}^{21} \\ Y_{22}^{22} = Y_{22}^{11} + \frac{\alpha}{\pi_K} Y_{22}^{21} \end{cases} \quad (8)$$

$$\begin{aligned} & \lambda_0 Y_{22} \lambda_0^{-1} Y_{11} Y_{22} + Y_{12} Y_{23} Y_{31} + Y_{13} Y_{32} Y_{21} \\ & - \lambda_0 Y_{22} \lambda_0^{-1} Y_{31} Y_{13} - Y_{11} Y_{32} Y_{23} - Y_{33} Y_{12} Y_{21} = 1 \end{aligned} \quad (9)$$

2. If  $i_0$  is even

$$\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \begin{pmatrix} \tau_{i_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau_{i_0} \end{pmatrix} \begin{pmatrix} \bar{Z}_{33} & \bar{Z}_{23} & \bar{Z}_{13} \\ \bar{Z}_{32} & \bar{Z}_{22} & \bar{Z}_{12} \\ \bar{Z}_{31} & \bar{Z}_{21} & \bar{Z}_{11} \end{pmatrix} = \begin{pmatrix} \tau_{i_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau_{i_0} \end{pmatrix} \quad (10)$$

$$\begin{pmatrix} \bar{Z}_{33} & \bar{Z}_{23} & \bar{Z}_{13} \\ \bar{Z}_{32} & \bar{Z}_{22} & \bar{Z}_{12} \\ \bar{Z}_{31} & \bar{Z}_{21} & \bar{Z}_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{i_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{i_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (11)$$

$$\frac{2}{\pi_K^{i_0}}(Z_{31}^{11}Z_{11}^{11} + \beta Z_{31}^{21}Z_{11}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(Z_{31}^{21}Z_{11}^{11} + Z_{11}^{21}Z_{31}^{11}) = -(\bar{Z}_{21}Z_{21})^{11} \quad (12)$$

$$\frac{2}{\pi_K^{i_0}}(Z_{13}^{11}Z_{33}^{11} + \beta Z_{13}^{21}Z_{33}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(Z_{13}^{21}Z_{33}^{11} + Z_{33}^{21}Z_{13}^{11}) = -(\bar{Z}_{23}Z_{23})^{11} \quad (13)$$

$$\text{for all } i, j \in \{1, 2, 3\}, \begin{cases} Z_{ij}^{12} = -\beta Z_{ij}^{21} \\ Z_{ij}^{22} = Z_{ij}^{11} + \alpha Z_{ij}^{21} \end{cases} \quad (14)$$

$$\sum_{\sigma \in \text{Sym}(3)} [(-1)^{\text{sgn}(\sigma)} \prod_{i=1}^3 Z_{i\sigma(i)}] = 1 \quad (15)$$

*Proof.* Recall that we only write down the case  $i_0$  odd. Let  $I$  be the ideal in  $\mathcal{O}_K[\mathbf{A}^{36}]$  generated by those equations. We want to show that  $I = \ker \varphi_{i_0}$  (see Definition 4.19).

**Claim 1.**  $I \leq \ker \varphi_{i_0}$

*Proof of the claim:* This can easily be checked equation by equation. For example,  $\varphi_{i_0}(\lambda_{n_0} Y_{21}) = X_{21}$ , hence  $\varphi_{i_0}(\bar{Y}_{21} v_{n_0}) = \bar{X}_{21}$ . Hence,  $\varphi_{i_0}^{-1}$  of the equalities

$$\begin{pmatrix} \bar{X}_{33} & \bar{X}_{23} & \bar{X}_{13} \\ \bar{X}_{32} & \bar{X}_{22} & \bar{X}_{12} \\ \bar{X}_{31} & \bar{X}_{21} & \bar{X}_{11} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in  $K[\text{SU}_3]$  gives the equalities

$$\begin{pmatrix} \bar{Y}_{33} & \bar{Y}_{23} v_{n_0} & \bar{Y}_{13} \\ v_{n_0}^{-1} \bar{Y}_{32} & v_{n_0}^{-1} \bar{Y}_{22} v_{n_0} & v_{n_0}^{-1} \bar{Y}_{12} \\ \bar{Y}_{31} & \bar{Y}_{21} v_{n_0} & \bar{Y}_{11} \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \lambda_{n_0}^{-1} & Y_{13} \\ \lambda_{n_0} Y_{21} & \lambda_{n_0} Y_{22} \lambda_{n_0}^{-1} & \lambda_{n_0} Y_{23} \\ Y_{31} & Y_{32} \lambda_{n_0}^{-1} & Y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, multiplying with  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_0^{-1} \tau_{n_0+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$  on the left and with  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{n_0+1} v_0^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$  on the right, we get

$$\begin{pmatrix} \bar{Y}_{33} & \bar{Y}_{23} v_{n_0} & \bar{Y}_{13} \\ \bar{Y}_{32} & \bar{Y}_{22} v_{n_0} & \bar{Y}_{12} \\ \bar{Y}_{31} & \bar{Y}_{21} v_{n_0} & \bar{Y}_{11} \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ \lambda_{n_0} Y_{21} & \lambda_{n_0} Y_{22} & \lambda_{n_0} Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ which is Equation 2.}$$



As another example, from Equation 2, we get in particular

$$\overline{Y}_{33}Y_{13} + \tau_{i_0}\overline{Y}_{23}Y_{23} + \overline{Y}_{13}Y_{33} = 0 \quad (2, L1C3)$$

The 11 component of Equation 2, L1C3 reads

$$Y_{33}^{22}Y_{13}^{11} - Y_{33}^{12}Y_{13}^{21} + Y_{13}^{22}Y_{33}^{11} - Y_{13}^{12}Y_{33}^{21} = -\tau_{i_0}(\overline{Y}_{23}Y_{23})^{11}$$

Now, using  $Y_{33}^{22} = Y_{33}^{11} + \alpha Y_{33}^{21}$  and  $Y_{33}^{12} = -\beta Y_{33}^{21}$  (and similarly for  $Y_{13}$ ), we get

$$2(Y_{33}^{11}Y_{13}^{11} + \beta Y_{33}^{21}Y_{13}^{21}) + \alpha(Y_{33}^{21}Y_{13}^{11} + Y_{13}^{21}Y_{33}^{11}) = -\tau_{i_0}(\overline{Y}_{23}Y_{23})^{11}$$

But note that all coefficients in this equation have valuation greater than or equal to  $\omega(\pi_K^{i_0})$

(because  $\omega(\pi_K^{i_0}) = \begin{cases} \omega(2) & \text{if } \alpha = 0 \\ \omega(\alpha) & \text{if } \alpha \neq 0 \end{cases}$  by Definition 4.5, and if  $\alpha \neq 0$ ,  $\omega(\alpha) \leq \omega(2)$  by Lemma 3.3).

Hence we can divide both sides by  $\pi_K^{i_0}$  and still have an equation with coefficients in  $\mathcal{O}_K$ . Checking the other equations is a similar task. ■

Let  $\mathfrak{Y}$  be the closed  $\mathcal{O}_K$ -subscheme of  $\mathbf{A}^{36}$  defined by the ideal  $I$ . By Claim 1,  $\underline{\text{SU}}_3$  is a closed subscheme of  $\mathfrak{Y}$ , and we want to prove that they are equal. The crux of the proof relies on investigating the closed fibre of the  $\mathcal{O}_K$ -scheme  $\mathfrak{Y}$ , or in other words, the scheme  $\mathfrak{Y}_{\overline{K}}$  over  $\overline{K}$ . As it will be needed later, we elucidate what  $\mathfrak{Y}_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}$  looks like, and then deduce what we want about  $\mathfrak{Y}_{\overline{K}}$ .

**Claim 2.**  $\mathfrak{Y}_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}$  is the following  $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$ -algebraic group: for any  $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$ -algebra  $R$ ,

$$\begin{aligned} \mathfrak{Y}_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}(R) = \left\{ \begin{pmatrix} \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{21}^{11} & w_{11}^{11} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{23}^{11} & w_{13}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{21}^{11} & -\frac{\beta}{\pi_K} w_{21}^{21} \\ w_{21}^{21} & \pi_K w_{21}^{11} \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} w_{23}^{11} & -\frac{\beta}{\pi_K} w_{23}^{21} \\ w_{23}^{21} & \pi_K w_{23}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{31}^{11} & -\beta w_{31}^{21} \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} w_{33}^{11} & -\beta w_{33}^{21} \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \end{pmatrix} \mid w_{ij} \in R; w_{11}w_{33} - w_{13}w_{31} = 1 \right. \\ \left. \frac{2}{\pi_K^{i_0}}(w_{31}^{11}w_{11}^{11} + \beta w_{31}^{21}w_{11}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(w_{31}^{21}w_{11}^{11} + w_{11}^{21}w_{31}^{11}) = -(\overline{w}_{21}w_{21})^{11} \right. \\ \left. \frac{2}{\pi_K^{i_0}}(w_{13}^{11}w_{33}^{11} + \beta w_{13}^{21}w_{33}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(w_{13}^{21}w_{33}^{11} + w_{33}^{21}w_{13}^{11}) = -(\overline{w}_{23}w_{23})^{11} \right\} \quad (*) \end{aligned}$$

where the group structure is the one coming from the representation of elements as forming a  $3 \times 3$  matrix.

*Proof of the claim:* We have to analyse our equations modulo  $\pi_K^{i_0}$ , or in other words, work in the ring  $(\mathcal{O}_K/\mathfrak{m}_K^{i_0})[Y_{ij}^{kl}]/I$ . In particular, in view of Definition 4.5 and Lemma 3.3, we are now working in characteristic 2.

From Equation 1, we get  $Y_{22}\overline{Y}_{22} = 1$ , so that in particular,  $Y_{22}$  and  $\overline{Y}_{22}$  are invertible matrices. Still from Equation 1, we also have  $Y_{22}\overline{Y}_{32} = 0 = Y_{22}\overline{Y}_{12}$ . Hence,  $\overline{Y}_{32} = 0 = \overline{Y}_{12}$ , so that also  $Y_{32} = 0 = Y_{12}$ . This implies that Equation 9 simplifies to

$$\lambda_0 Y_{22} \lambda_0^{-1} (Y_{11}Y_{33} - Y_{31}Y_{13}) = 1$$

On the other hand, Equation 2 gives  $\overline{Y}_{33}Y_{11} + \overline{Y}_{13}Y_{31} = 1$ . But  $\overline{Y}_{33} = Y_{33}$  and  $\overline{Y}_{13} = Y_{13}$  (which follows from Equation 5 and the fact that the characteristic is 2). Hence, we conclude that  $\lambda_0 Y_{22} \lambda_0^{-1} = 1$ , and hence that  $Y_{22} = 1$  (using Equation 8). Combining what we know so far with Equations 5 and 6, we get the claim. ■

Let  $\mathcal{RSL}_2$  be the Weil restriction from  $\mathcal{O}_L/\mathfrak{m}_L^{2i_0}$  to  $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$  of the algebraic group  $\text{SL}_2$ . In more concrete terms, for any  $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$ -algebra  $R$ ,

$$\mathcal{RSL}_2(R) = \left\{ \begin{pmatrix} \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{21}^{11} & w_{11}^{11} \end{pmatrix} & \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{23}^{11} & w_{13}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{31}^{11} & -\beta w_{31}^{21} \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} & \begin{pmatrix} w_{33}^{11} & -\beta w_{33}^{21} \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \end{pmatrix} \mid w_{ij} \in R \right\}$$

**Claim 3.** For any  $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$ -algebra  $R$ , there exists a (functorial in  $R$ ) group homomorphism

$$f_R: \mathfrak{Y}_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}(R) \rightarrow \mathcal{R}\mathrm{SL}_2(R)$$

$$\left( \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \right) \mapsto \left( \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \right)$$

$$\left( \begin{pmatrix} w_{21}^{11} & -\frac{\beta}{\pi_K} w_{21}^{21} \\ w_{21}^{21} & \pi_K w_{21}^{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_{23}^{11} & -\frac{\beta}{\pi_K} w_{23}^{21} \\ w_{23}^{21} & \pi_K w_{23}^{11} \end{pmatrix} \right) \mapsto \left( \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \right)$$

$$\left( \begin{pmatrix} w_{31}^{11} & -\beta w_{31}^{21} \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{33}^{11} & -\beta w_{33}^{21} \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \right) \mapsto \left( \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \right)$$

Furthermore,  $f_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}$  is surjective.

*Proof of the claim:* The map  $f_R$  is readily seen to be a group homomorphism. Let us check that  $f_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}$  is surjective. On the left hand side, apart from a determinant-like equation, we have also equations like  $\frac{2}{\pi_K^{i_0}}(w_{31}^{11}w_{11}^{11} + \beta w_{31}^{21}w_{11}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(w_{31}^{21}w_{11}^{11} + w_{11}^{21}w_{31}^{11}) = -(\overline{w}_{21}w_{21})^{11}$ . But note that  $(\overline{w}_{21}w_{21})^{11} = \pi_K(w_{21}^{11})^2 + (w_{21}^{21})^2$ . Since squaring is a surjective map on  $\mathcal{O}_K/\mathfrak{m}_K$ , we see that for any  $x \in \mathcal{O}_K/\mathfrak{m}_K^{i_0}$ , there exists  $w_{21}^{11}, w_{21}^{21} \in \mathcal{O}_K/\mathfrak{m}_K^{i_0}$  such that  $x = \pi_K(w_{21}^{11})^2 + (w_{21}^{21})^2$ . The surjectivity follows. ■

**Claim 4.** The  $\overline{K}$ -group  $\mathfrak{Y}_{\overline{K}}$  is smooth of dimension 8, and is connected.

*Proof of the claim:* Smoothness and the computation of the dimension over  $\overline{K}$  follows directly from Claim 2. Indeed, one can see directly that the dimension is in between 8 and 12. But the tangent space at the identity is obviously of codimension 4 in  $\mathbf{A}_{\overline{K}}^{12}$ , as wanted.

We now prove that  $\mathfrak{Y}_{\overline{K}}$  is connected. We have a morphism

$$f: \mathfrak{Y}_{\overline{K}} \rightarrow (\mathrm{SL}_2)_{\overline{K}}: \left( \begin{pmatrix} w_{11}^{11} & 0 \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{13}^{11} & 0 \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \right) \mapsto \begin{pmatrix} w_{11}^{11} & w_{13}^{11} \\ w_{31}^{11} & w_{33}^{11} \end{pmatrix}$$

$$\left( \begin{pmatrix} w_{21}^{11} & w_{21}^{21} \\ w_{21}^{21} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_{23}^{11} & w_{23}^{21} \\ w_{23}^{21} & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} w_{11}^{11} & w_{13}^{11} \\ w_{31}^{11} & w_{33}^{11} \end{pmatrix}$$

$$\left( \begin{pmatrix} w_{31}^{11} & 0 \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{33}^{11} & 0 \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \right) \mapsto \begin{pmatrix} w_{11}^{11} & w_{13}^{11} \\ w_{31}^{11} & w_{33}^{11} \end{pmatrix}$$

which is surjective. The kernel is

$$\left\{ \left( \begin{pmatrix} 1 & 0 \\ w_{21}^{11} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ w_{21}^{21} & 0 \end{pmatrix} \right) \mid w_{11}^{21} + w_{33}^{21} = 0, (w_{21}^{21})^2 = \frac{\alpha}{\pi_K^{i_0}} w_{31}^{21}, (w_{23}^{21})^2 = \frac{\alpha}{\pi_K^{i_0}} w_{13}^{21} \right\}$$

which is clearly a product of connected schemes, hence is connected. So  $f: \mathfrak{Y}_{\overline{K}} \rightarrow (\mathrm{SL}_2)_{\overline{K}}$  is a surjective morphism, whose kernel and image are connected. Hence,  $\mathfrak{Y}_{\overline{K}}$  is connected. ■

**Claim 5.** We have  $(\underline{\mathrm{SU}}_3)_K = \mathfrak{Y}_K$ , and  $(\underline{\mathrm{SU}}_3)_{\overline{K}} = \mathfrak{Y}_{\overline{K}}$ .

*Proof of the claim:* Over  $K$ , we have a composition of closed embeddings  $\mathrm{SU}_3 \hookrightarrow (\underline{\mathrm{SU}}_3)_K \hookrightarrow \mathfrak{Y}_K$ . But  $\mathrm{SU}_3 \hookrightarrow \mathfrak{Y}_K$  is clearly an isomorphism, hence the claim. We now prove the equality of the closed fibre, i.e.  $(\underline{\mathrm{SU}}_3)_{\overline{K}} = \mathfrak{Y}_{\overline{K}}$ . Our argument is based on [GE]. By Claim 4,  $\mathfrak{Y}_{\overline{K}}$  is a smooth irreducible affine  $\overline{K}$ -scheme of dimension 8. But  $(\underline{\mathrm{SU}}_3)_{\overline{K}}$  is a closed subscheme of  $\mathfrak{Y}_{\overline{K}}$  of the same dimension. Hence  $(\underline{\mathrm{SU}}_3)_{\overline{K}} = \mathfrak{Y}_{\overline{K}}$ . Indeed, the  $\overline{K}$ -group scheme  $\mathfrak{Y}_{\overline{K}}$  is connected (respectively smooth) by Claim 4, thus irreducible (respectively reduced). Hence  $\overline{K}[\mathfrak{Y}_{\overline{K}}]$  is a domain. But the kernel of  $\overline{K}[\mathfrak{Y}_{\overline{K}}] \rightarrow \overline{K}[(\underline{\mathrm{SU}}_3)_{\overline{K}}]$  is contained in the nilradical of  $\overline{K}[\mathfrak{Y}_{\overline{K}}]$  (by Krull's principal ideal theorem), which shows that  $\overline{K}[\mathfrak{Y}_{\overline{K}}] \rightarrow \overline{K}[(\underline{\mathrm{SU}}_3)_{\overline{K}}]$  is injective as well (because being a domain,  $\overline{K}[\mathfrak{Y}_{\overline{K}}]$  has in particular a trivial nilradical). ■

We can now conclude the proof of Lemma 4.22. We have some closed embeddings  $\underline{\mathrm{SU}}_3 \hookrightarrow \mathfrak{Y} \hookrightarrow \mathbf{A}^{36}$ , and by Claim 5,  $\underline{\mathrm{SU}}_3 \hookrightarrow \mathfrak{Y}$  is an equality on fibres. Hence, since  $(\underline{\mathrm{SU}}_3)_K$  is dense in

$\underline{\mathrm{SU}}_3 = (\underline{\mathrm{SU}}_3)_K \sqcup (\underline{\mathrm{SU}}_3)_{\overline{K}}$  (because  $\underline{\mathrm{SU}}_3$  is a schematic adherence, see [BT84a, 1.2.6]), we conclude that  $\mathfrak{Y}_K$  is dense in  $\mathfrak{Y} = \mathfrak{Y}_K \sqcup \mathfrak{Y}_{\overline{K}}$  as well. In the terminology of [BT84a, 1.2.3], this precisely means that  $\mathfrak{Y}$  is without torsion. But there is a 1 – 1 correspondence between closed  $K$ -schemes of  $\mathbf{A}_K^{36}$  and closed  $\mathcal{O}_K$ -schemes of  $\mathbf{A}_{\mathcal{O}_K}^{36}$  without torsion ([BT84a, 1.2.6]). Since  $(\underline{\mathrm{SU}}_3)_K = \mathfrak{Y}_K$ , this concludes the proof.  $\square$

For  $G = \mathrm{SL}_2$  or  $\mathrm{SU}_3$ , we have just defined an integral model  $\underline{G}$ . We now check that in each case,  $\underline{G}(\mathcal{O}_K) \cong P_0$ .

**Lemma 4.23.** 1.  $\underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K) \cong P_0$

2. When  $\gamma = 0$ ,  $\underline{\mathrm{SU}}_3(\mathcal{O}_K) \cong P_0$

3. When  $\gamma > 0$ ,  $\underline{\mathrm{SU}}_3(\mathcal{O}_K) \cong P_0$

*Proof.* 1. When  $D = K$ , by definition,  $\underline{\mathrm{SL}}_2(\mathcal{O}_K) = \mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SL}}_2], \mathcal{O}_K)$ , which is clearly isomorphic to  $\mathrm{SL}_2(\mathcal{O}_K)$ . The case  $\underline{\mathrm{SL}}_{2,D}$  when  $[D : K] > 1$  is done in the appendix (see Lemma C.5).

2. Using the fact that  $\mathcal{O}_L \cong \mathcal{O}_K \oplus t \cdot \mathcal{O}_K$  (where  $t \in \mathcal{O}_L$  is as in Lemma 3.3), one can check that  $\mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SU}}_3], \mathcal{O}_K) \cong \{g \in \mathrm{SL}_3(\mathcal{O}_L) \mid {}^S g g = \mathrm{Id}\}$ , as wanted.

3. This has already been proved in Lemma 4.21.  $\square$

We now spell out what the group  $\underline{G}(\mathcal{O}_K/\mathfrak{m}_K^r)$  is, together with the homomorphism  $p_r: P_0 \rightarrow P_0^{0,r}$ .

**Lemma 4.24.** 1.  $\underline{\mathrm{SL}}_2(\mathcal{O}_K/\mathfrak{m}_K^r) \cong P_0^{0,r}$ . *Following the identifications*

$$\begin{array}{ccc} \underline{\mathrm{SL}}_2(\mathcal{O}_K) & \cong & \mathrm{SL}_2(\mathcal{O}_K) = P_0 \\ \downarrow & & \downarrow \\ \underline{\mathrm{SL}}_2(\mathcal{O}_K/\mathfrak{m}_K^r) & \cong & \mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r) = P_0^{0,r} \end{array}$$

the homomorphism  $p_r: P_0 \rightarrow P_0^{0,r}$  is the one induced by the projection of the coefficients  $\mathcal{O}_K \rightarrow \mathcal{O}_K/\mathfrak{m}_K^r$ .

2. More generally, for  $D$  a central division algebra of degree  $d$  over  $K$ ,  $\underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K/\mathfrak{m}_K^r) \cong P_0^{0,rd}$ . *Following the identifications*

$$\begin{array}{ccc} \underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K) & \cong & \mathrm{SL}_2(\mathcal{O}_D) = P_0 \\ \downarrow & & \downarrow \\ \underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K/\mathfrak{m}_K^r) & \cong & \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) = P_0^{0,rd} \end{array}$$

the homomorphism  $p_{rd}: P_0 \rightarrow P_0^{0,rd}$  is the one induced by the projection of the coefficients  $\mathcal{O}_D \rightarrow \mathcal{O}_D/\mathfrak{m}_D^{rd}$ .

3. When  $\gamma = 0$ , let  $\epsilon = \begin{cases} 1 & \text{if } L \text{ is unramified} \\ 2 & \text{if } L \text{ is ramified} \end{cases}$ . Then  $\underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^r) \cong P_0^{0,\epsilon r}$ . *Following the identifications*

$$\begin{array}{ccc} \underline{\mathrm{SU}}_3(\mathcal{O}_K) & \cong & P_0 \leq \mathrm{SL}_3(\mathcal{O}_L) \\ \downarrow & & \downarrow \\ \underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^r) & \cong & P_0^{0,\epsilon r} \leq \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^{\epsilon r}) \end{array}$$

the homomorphism  $p_{\epsilon^r}: P_0 \rightarrow P_0^{0,\epsilon^r}$  is the one induced by the projection of the coefficients  $\mathcal{O}_L \rightarrow \mathcal{O}_L/\mathfrak{m}_L^{\epsilon^r}$ .

4. When  $\gamma > 0$ , there exists a surjective homomorphism  $\underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^{i_0}) \twoheadrightarrow P_0^{0,2i_0}$ . For  $r \leq 2i_0$ , we thus have the following diagram

$$\begin{array}{ccc} \underline{\mathrm{SU}}_3(\mathcal{O}_K) & \cong & P_0 \leq \mathrm{SL}_3(L) \\ \downarrow & & \downarrow f_1 \\ \underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^{i_0}) & \twoheadrightarrow & P_0^{0,2i_0} = \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^{2i_0}) \\ & & \downarrow f_2 \\ & & P_0^{0,r} = \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \end{array}$$

where  $f_2$  is induced by the ring homomorphism  $\mathcal{O}_L/\mathfrak{m}_L^{2i_0} \rightarrow \mathcal{O}_L/\mathfrak{m}_L^r$ . The resulting homomorphism  $p_r: P_0 \rightarrow P_0^{0,r}$  is given by the following formula:

$$\begin{aligned} f_2 \circ f_1: P_0 \leq \mathrm{SL}_3(L) &\rightarrow \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \\ \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} &\mapsto \begin{pmatrix} p(g_{11}) & p(g_{13}) \\ p(g_{31}) & p(g_{33}) \end{pmatrix} \end{aligned}$$

(where  $p: \mathcal{O}_L \rightarrow \mathcal{O}_L/\mathfrak{m}_L^r$  denotes the projection modulo  $\mathfrak{m}_L^r$ ).

*Proof.* 1. By definition,  $\underline{\mathrm{SL}}_2(\mathcal{O}_K/\mathfrak{m}_K^r) = \mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SL}}_2], \mathcal{O}_K/\mathfrak{m}_K^r)$ , which is clearly isomorphic to  $\mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r)$ , as wanted.

2. This is treated in the appendix (see Lemma C.5).

3. Using the fact that if  $L$  is unramified (respectively  $L$  is ramified),  $\mathcal{O}_L/\mathfrak{m}_L^r \cong \mathcal{O}_K/\mathfrak{m}_K^r \oplus t \cdot \mathcal{O}_K/\mathfrak{m}_K^r$  (respectively  $\mathcal{O}_L/\mathfrak{m}_L^{2r} \cong \mathcal{O}_K/\mathfrak{m}_K^r \oplus t \cdot \mathcal{O}_K/\mathfrak{m}_K^r$ ), one can check that

$$\begin{aligned} \mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SU}}_3], \mathcal{O}_K/\mathfrak{m}_K^r) &\cong \{g \in \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^r) \mid {}^S \bar{g}g = \mathrm{Id}\} \\ (\text{respectively } \mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SU}}_3], \mathcal{O}_K/\mathfrak{m}_K^r) &\cong \{g \in \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^{2r}) \mid {}^S \bar{g}g = \mathrm{Id}\}) \end{aligned}$$

as wanted.

4. Recall the definition of the Weil restriction  $\mathcal{R}\mathrm{SL}_2$  of  $\mathrm{SL}_2$  from  $\mathcal{O}_L/\mathfrak{m}_L^{2i_0}$  to  $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$  that we discussed before Claim 3 in the proof of Lemma 4.22. Note that  $\mathcal{R}\mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^{i_0}) \cong \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^{2i_0})$ . Now, the existence of a surjective homomorphism  $\underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^{i_0}) \rightarrow P_0^{0,r}$  was proved in Claim 3.  $\square$

Our work on integral models, and especially the fact that they are smooth schemes over  $\mathcal{O}_K$ , allows us to deduce the surjectivity of  $P_0 \rightarrow P_0^{0,r}$ . For this, we use a well-known generalised version of Hensel's lemma for smooth schemes, that we now recall.

**Theorem 4.25** (Hensel's lemma for smooth schemes). *Let  $X$  be a smooth  $\mathcal{O}_K$ -scheme, and let  $r_1 \geq r_2 \in \mathbf{N} \cup \{\infty\}$ . Then  $X(\mathcal{O}_K/\mathfrak{m}_K^{r_1}) \rightarrow X(\mathcal{O}_K/\mathfrak{m}_K^{r_2})$  is surjective (where by convention,  $\mathfrak{m}_K^\infty = (0)$ ).*

*Proof.* It suffices to prove that for all  $r \in \mathbf{N}$ ,  $X(\mathcal{O}_K) \rightarrow X(\mathcal{O}_K/\mathfrak{m}_K^r)$  is surjective. For  $r = 1$ , this is [Gro67, Théorème 18.5.17]. In the general case, note that as remarked below [Gro67, Définition 18.5.5],  $(S, S_0)$  is a Henselian couple if and only if  $(S_{red}, (S_0)_{red})$  is so. We deduce that  $(\mathrm{Spec} \mathcal{O}_K, \mathrm{Spec} \mathcal{O}_K/\mathfrak{m}_K^r)$  is a Henselian couple. Thus the proof of Théorème 18.5.17 applies verbatim to our situation, upon making one change: replace the reference to 18.5.11(b) to a reference to 18.5.4(b) (taking  $S = \mathrm{Spec} \mathcal{O}_K$  and  $S_0 = \mathrm{Spec} \mathcal{O}_K/\mathfrak{m}_K^r$  in the notation of 18.5.4).  $\square$

**Corollary 4.26.** 1. In the  $\mathrm{SL}_2(D)$  case, let  $d$  be the degree of  $D$  over  $K$ . The map  $p_r: P_0 \rightarrow P_0^{0,rd}$  is surjective, for all  $r \in \mathbf{N}$ .

2. In the  $\mathrm{SU}_3$  case when  $\gamma = 0$ , the map  $p_{\epsilon r}: P_0 \rightarrow P_0^{0,\epsilon r}$  is surjective, for all  $r \in \mathbf{N}$  (where  $\epsilon = 1$  if  $L$  is unramified, and  $\epsilon = 2$  if  $L$  is ramified).

3. In the  $\mathrm{SU}_3$  case when  $\gamma > 0$ , the map  $p_r: P_0 \rightarrow P_0^{0,r}$  is surjective, for all  $r \in \mathbf{N}$  such that  $r \leq 2i_0$ .

*Proof.* In each case, this is a direct consequence of the commutative square involving  $P_0 \rightarrow P_0^{0,r}$  given in Lemma 4.24, together with the fact that the integral model is smooth, so that Theorem 4.25 applies to the left hand side of the diagram.

In the  $\mathrm{SU}_3$  case when  $\gamma > 0$ , we furthermore have to argue that the map  $f_2$  appearing in Lemma 4.24 is surjective, but this is just another instance of Hensel's Lemma (Theorem 4.25) in the  $\mathrm{SL}_2$  case.  $\square$

**Remark 4.27.** In the  $\mathrm{SU}_3$  case when  $\gamma = 0$  and  $L$  is ramified, we did not prove that the map  $p_r: P_0 \rightarrow P_0^{0,r}$  is surjective when  $r$  is odd. We did not take the time to investigate further whether such a surjectivity holds, since we do not need it.

Along with the surjectivity of the restriction map  $p_r: P_0 \rightarrow P_0^{0,r}$ , one of the key result in our local description of the ball of radius  $r$  is that  $p_r$  is also somehow injective enough. This result can be seen as a natural generalisation of [BT84a, Corollaire 4.6.8].

**Lemma 4.28.** Let  $r \in \mathbf{N}$ .

1. In the  $\mathrm{SL}_2(D)$  case, let  $x \in [-\omega(\pi_K^{rd}), \omega(\pi_K^{rd})]$ . Then  $p_{rd}^{-1}(P_x^{0,rd}) \subset P_x$  (where  $d$  is the degree of  $D$  over  $K$ ).
2. In the  $\mathrm{SU}_3$  case when  $\gamma = 0$ , let  $x \in [-\omega(\pi_L^{\epsilon r}), \omega(\pi_L^{\epsilon r})]$ . Then  $p_{\epsilon r}^{-1}(P_x^{0,\epsilon r}) \subset P_x$  (where  $\epsilon = 1$  if  $L$  is unramified, and  $\epsilon = 2$  if  $L$  is ramified).
3. In the  $\mathrm{SU}_3$  case when  $\gamma > 0$ , assume that  $r \leq 2i_0$ , and let  $x \in [-\omega(\pi_L^r), \omega(\pi_L^r)]$ . Then  $p_r^{-1}(P_x^{0,r}) \subset P_x$ .

*Proof.* In the  $\mathrm{SL}_2(D)$  case (respectively the  $\mathrm{SU}_3$  case when  $\gamma = 0$ ), belonging to  $p_{rd}^{-1}(P_x^{0,rd})$  (taking  $d = 1$  in the  $\mathrm{SU}_3$  case) implies that the valuation of the off diagonal entries are big enough. Hence, the result follows directly from Definition 3.2 and Definition 3.7.

In the  $\mathrm{SU}_3$  case when  $\gamma > 0$ , let  $g \in p_r^{-1}(P_x^{0,r})$ . We want to show that  $g \in P_x$ . We can assume that  $x \in [0, \omega(\pi_L^r)]$ , the argument when  $x$  is negative being similar.

By assumption, we know that  $\omega(g_{31}) \geq x$ , and we want to show that this implies  $\omega(g_{21}) \geq \frac{x}{2} + \gamma$  and  $\omega(g_{32}) \geq \frac{x}{2} - \gamma$ . Since  $g \in \mathrm{SU}_3(K)$ ,  $\varphi_{i_0}(g) \in \underline{\mathrm{SU}}_3(\mathcal{O}_K)$ . In particular, the coefficients of  $g$  satisfy

$$\frac{2}{\pi_K^{i_0}}(g_{31}^{11}g_{11}^{11} + \beta g_{31}^{21}g_{11}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(g_{31}^{21}g_{11}^{11} + g_{11}^{21}g_{31}^{11}) = -(\bar{g}_{21}\tau_{i_0}^{-1}g_{21})^{11}$$

Note that  $(\bar{g}_{21}g_{21})^{11} = (g_{21}^{11} + \alpha g_{21}^{21})g_{21}^{11} + \beta(g_{21}^{21})^2$ , and that  $\tau_{i_0}^{-1}$  is just multiplication by  $\pi_K^{-i_0}$ . Also recall that if  $\alpha = 0$ ,  $\omega(2) = \omega(\pi_K^{i_0}) = 2\gamma$ , while if  $\alpha \neq 0$ ,  $\omega(\alpha) = \omega(\pi_K^{i_0}) = 2\gamma + \omega(\pi_L)$  (see Definition 3.5 and Definition 4.5). Furthermore, if  $\alpha \neq 0$ ,  $\omega(\alpha) \leq \omega(2)$  by Lemma 3.3. Hence, we get

$$\omega((g_{21}^{11} + \alpha g_{21}^{21})g_{21}^{11} + \beta(g_{21}^{21})^2) \geq 2\gamma + x$$

But  $\omega((g_{21}^{11} + \alpha g_{21}^{21})g_{21}^{11} + \beta(g_{21}^{21})^2) = \min\{\omega((g_{21}^{11})^2); \omega(\beta(g_{21}^{21})^2)\} = 2\omega(g_{21})$ , so that  $\omega(g_{21}) \geq \gamma + \frac{x}{2}$ , as wanted.

Finally, using again that  $g \in \mathrm{SU}_3(K)$ , we also find  $g_{21}\bar{g}_{33} + g_{22}\bar{g}_{32} + g_{23}\bar{g}_{31} = 0$ . By Claim 2 of Lemma 4.22, if  $i_0$  is odd (respectively even)  $\lambda_{n_0}^{-1}g_{22}\lambda_{n_0}$  (respectively  $g_{22}$ ) is equal to  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  modulo  $\mathfrak{m}_K^{i_0}$ . Thus  $g_{22}$  is in particular of valuation 0. Hence, we get that  $\bar{g}_{32}$  is of the same valuation than  $g_{21}\bar{g}_{33} + g_{23}\bar{g}_{31}$ . Since  $g \in P_0$ ,  $\omega(g_{33}) \geq 0 \leq \omega(g_{23})$ , and we know that  $\omega(g_{31}) \geq x \geq \frac{x}{2} \leq \omega(g_{21})$ . This concludes the proof.  $\square$

We arrive finally at our main result, which is that the ball of radius  $rd$  (respectively  $r$ ), together with the action of  $\mathrm{SL}_2(\mathcal{O}_D)$  (respectively  $\mathrm{SU}_3(\mathcal{O}_K)$ ), is encoded in  $P_0^{0,rd}$  (respectively  $P_0^{0,r}$ ). Let us first state our definition of the ball of radius  $r$ .

**Definition 4.29.** Let  $G$  be  $\mathrm{SL}_2(D)$  (respectively  $\mathrm{SU}_3^{L/K}(K)$ ). Let  $\pi = \pi_D$  and  $d = \sqrt{[D:K]}$  (respectively  $\pi = \pi_L$  and  $d = 1$ ). The ball of radius  $rd$  around  $0 \in \mathbf{R}$  is

$$B_0(rd) = \{[(g, x)]_0 \in \mathcal{I}_0 \mid x \in [-\omega(\pi^{rd}), \omega(\pi^{rd})] \subset \mathbf{R}, g \in P_0\}$$

**Remark 4.30.** Recall that by Lemma 3.16, the map  $B_0(rd) \rightarrow \mathcal{I}: [(g, x)]_0 \mapsto [(g, x)]$  is an equivariant embedding.

The following result explains why we call  $B_0(rd)$  the ball of radius  $r$ .

**Lemma 4.31.** In the  $\mathrm{SL}_2(D)$  case (respectively the  $\mathrm{SU}_3(K)$  case), let  $\pi = \pi_D$  and  $d = \sqrt{[D:K]}$  (respectively  $\pi = \pi_L$  and  $d = 1$ ). Let us identify  $B_0(rd)$  with its image in  $\mathcal{I}$  under  $\mathcal{I}_0 \rightarrow \mathcal{I}$ . Renormalise the distance on  $\mathbf{R}$  so that  $d_{\mathbf{R}}(0, \omega(\pi)) = 1$ , and put the metric  $d_{\mathcal{I}}$  on  $\mathcal{I}$  arising from the distance  $d_{\mathbf{R}}$  (see Remark 3.11). Then  $B_0(rd) = \{p \in \mathcal{I} \mid d_{\mathcal{I}}([\mathrm{Id}, 0], p) \leq rd\}$ .

*Proof.* Looking at the embedding  $\mathbf{R} \hookrightarrow \mathcal{I}: x \mapsto [(\mathrm{Id}, x)]$ , it is easy to identify which  $x \in \mathbf{R}$  are vertices of the tree  $\mathcal{I}$ . Indeed,  $x \in \mathbf{R}$  is a vertex of  $\mathcal{I}$  if and only if  $P_x$  strictly contains  $P_{x+\varepsilon}$  (where  $\varepsilon$  is a real number such that  $|\varepsilon| < \omega(\pi)$ ). From our description of  $P_x$ , one readily check that  $x \in \mathbf{R}$  is a vertex of  $\mathcal{I}$  if and only if  $x \in \mathbf{Z} \cdot \omega(\pi) = \omega(\pi^{\mathbf{Z}})$ . Now, if  $[(g, x)] \in B_0(rd)$ , then  $d_{\mathbf{R}}(0, x) \leq rd$  by our normalisation of the distance on  $\mathbf{R}$ , while if  $d_{\mathcal{I}}([\mathrm{Id}, 0], [(g, x)]) \leq rd$ , then  $(g, x) \sim (\mathrm{Id}, y)$  with  $d_{\mathbf{R}}(0, y) \leq rd$ , so that  $[(g, x)] \in B_0(rd)$ , as wanted.  $\square$

**Theorem 4.32.** Let  $r \in \mathbf{N}$ . Depending on cases, we assume the following:

1. In the  $\mathrm{SU}_3$  case when  $\gamma = 0$  and  $L$  is ramified, we assume that  $r$  is even.
2. In the  $\mathrm{SU}_3$  case when  $\gamma > 0$ , we assume that  $r \leq 2i_0$ .

Also, let  $d = \sqrt{[D:K]}$  in the  $\mathrm{SL}_2(D)$  case (respectively  $d = 1$  in the  $\mathrm{SU}_3$  case). The map  $B_0(rd) \rightarrow \mathcal{I}^{0,rd}: [(g, x)]_0 \mapsto [(p_{rd}(g), x)]^{0,rd}$  is a  $(p_{rd}: P_0 \rightarrow P_0^{0,rd})$ -equivariant bijection.

*Proof.* It is readily seen that the given map is well-defined.

- Injectivity: let  $[(g, x)]_0, [(h, y)]_0 \in B_0(rd)$  be such that they have the same image in  $\mathcal{I}^{0,rd}$ . By Remark 4.10, it means that for all  $\tilde{n} \in N^{0,rd}$  such that  $\nu(\tilde{n})(x) = y$ ,  $p_{rd}(g)^{-1}p_{rd}(h)\tilde{n} \in P_x^{0,rd}$ . So, we can assume that  $\tilde{n}$  is either equal to  $\mathrm{Id}$ , or is of the form  $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$  (respectively  $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ) in the  $\mathrm{SL}_2(D)$  case (respectively the  $\mathrm{SU}_3$  case). Hence, there exists  $n \in N$  such that  $p_{rd}(n) = \tilde{n}$ . But  $\nu(n)(x) = y$ , and  $g^{-1}hn \in p_{rd}^{-1}(P_x^{0,rd}) \subset P_x$  by Lemma 4.28. Hence,  $[(g, x)]_0 = [(h, y)]_0$ , as wanted.
- Surjectivity: follows directly from the surjectivity of  $p_{rd}: P_0 \rightarrow P_0^{0,rd}$  (Corollary 4.26).
- Equivariance:  $h \cdot [(g, x)]_0 = [(hg, x)]_0 \mapsto [(p_{rd}(hg), x)]^{0,rd} = p_{rd}(h) \cdot [(p_{rd}(g), x)]^{0,rd}$ .  $\square$

## 5 Convergences on the arithmetic side

### 5.1 The topological space of quadratic pairs of local fields

**Definition 5.1.** Consider the set of pairs of local fields  $(K, L)$  where either

1.  $K = L$  (equipped with the trivial conjugation action).
2.  $L$  is a separable quadratic extension of  $K$ .

We say that a pair  $(K, L)$  is trivial (respectively ramified, respectively unramified) if  $L = K$  (respectively  $L$  is quadratic ramified, respectively  $L$  is quadratic unramified). We also use those adjectives for  $L$ , when the pair under consideration is implicit. Furthermore, we freely amalgamate the notions of local fields and trivial pair of local fields.

**Remark 5.2.** Strictly speaking, a trivial extension of a local field is both ramified and unramified, but we nevertheless adopt the above vocabulary to be able to easily differentiate the three kinds of pairs.

**Definition 5.3.** We say that two pairs  $(K_1, L_1)$  and  $(K_2, L_2)$  are isomorphic if there exists a conjugation equivariant isomorphism between the two pairs. Let  $\mathcal{L}$  be the set of pairs of local fields as in Definition 5.1, up to isomorphism. For each prime  $p$ , let us also define  $\mathcal{L}_{p^n} = \{(K, L) \in \mathcal{L} \mid |\overline{K}| = p^n\}$ .

Following an idea dating back to Krasner (see [Del84] for references, this idea is also used in e.g. [Kaz86]), we define a metric on the space  $\mathcal{L}$ .

**Definition 5.4.** Let  $(K_1, L_1)$  and  $(K_2, L_2)$  be in  $\mathcal{L}$ . The conjugation induces an automorphism of  $\mathcal{O}_{L_i}/\mathfrak{m}_{L_i}^r$ , for any  $r \in \mathbf{N}$ . We say that  $(K_1, L_1)$  is  $r$ -close to  $(K_2, L_2)$  if and only if there exists a conjugation equivariant isomorphism  $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r \rightarrow \mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^r$  inducing an isomorphism  $\mathcal{O}_{K_1}/(\mathfrak{m}_{L_1}^r \cap \mathcal{O}_{K_1}) \rightarrow \mathcal{O}_{K_2}/(\mathfrak{m}_{L_2}^r \cap \mathcal{O}_{K_2})$ .

**Remark 5.5.** If  $L$  is unramified or if the residue characteristic is not 2, a conjugation equivariant isomorphism  $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r \rightarrow \mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^r$  always induces an isomorphism  $\mathcal{O}_{K_1}/(\mathfrak{m}_{L_1}^r \cap \mathcal{O}_{K_1}) \rightarrow \mathcal{O}_{K_2}/(\mathfrak{m}_{L_2}^r \cap \mathcal{O}_{K_2})$ , since in those cases,  $\mathcal{O}_{K_1}/(\mathfrak{m}_{L_1}^r \cap \mathcal{O}_{K_1})$  is the invariant subring of  $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r$ . We do not know if it still holds in the ramified and residue characteristic 2 case. Also note that  $(\mathfrak{m}_{L_1}^r \cap \mathcal{O}_{K_1}) = \begin{cases} \mathfrak{m}_{K_1}^{\lceil \frac{r}{2} \rceil} & \text{if } L_1 \text{ is ramified} \\ \mathfrak{m}_{K_1}^r & \text{if } L_1 \text{ is unramified} \end{cases}$ .

**Remark 5.6.** Note that being  $r$ -close is an equivalence relation, and that if  $r \geq l$  and  $(K_1, L_1)$  is  $r$ -close to  $(K_2, L_2)$ , then  $(K_1, L_1)$  is  $l$ -close to  $(K_2, L_2)$ .

We now observe that this notion of closeness induces a non-archimedean metric on  $\mathcal{L}$ . Let

$$d: \mathcal{L} \times \mathcal{L} \rightarrow \mathbf{R}_{\geq 0}: d((K_1, L_1), (K_2, L_2)) = \inf\left\{\frac{1}{2^r} \mid (K_1, L_1) \text{ is } r\text{-close to } (K_2, L_2)\right\}$$

**Lemma 5.7.**  $d(\cdot, \cdot)$  is a non-archimedean metric on  $\mathcal{L}$ .

*Proof.* If  $d((K_1, L_1), (K_2, L_2)) = 0$ , then  $\mathcal{O}_{L_1}$  and  $\mathcal{O}_{L_2}$  are equivariantly isomorphic. But then, the pairs of field of fraction are isomorphic in  $\mathcal{L}$ , as wanted. The fact that this distance is non-archimedean is a consequence of Remark 5.6.  $\square$

**Remark 5.8.** With this definition, if  $d((K_1, L_1), (K_2, L_2)) \leq \frac{1}{2}$ , then  $L_1$  is unramified if and only if  $L_2$  is unramified. In other words, unramified pairs are always at distance 1 from other kind of pairs. This is because  $L$  is unramified if and only if the conjugation action is non-trivial on the residue field.

A crucial fact about the space  $\mathcal{L}_{p^n}$  (for a fixed prime power  $p^n$ , as in Definition 5.3) is that it is a compact space. As was outlined in the introduction, this is one of the key observation to prove that  $\mathcal{S}_T^{\text{qs-alg}}$  is closed in  $\mathcal{S}_T$ . In fact, it is even possible to give an explicit description of the metric space  $\mathcal{L}_{p^n}$ . It takes some time to establish this explicit description, but it only uses basic facts from the theory of local fields. The corner stone in this description is Theorem 5.9 which is certainly well known to experts (this is for example used implicitly in [Kaz86]). While working on this paper, we learnt that it had also been obtained and used independently in [dlST15, Lemma 1.3]. Given its importance, we decide nevertheless to include our own proof.



**Theorem 5.9.** *Let  $K$  be a totally ramified extension of degree  $k$  of  $\mathbf{Q}_{p^n}$ . The distance between  $K$  and  $\mathbf{F}_{p^n}((X))$  is  $\frac{1}{2k}$ . More explicitly, let  $\{a_x\}_{x \in \mathbf{F}_{p^n}} \subset \mathbf{Q}_{p^n}$  be a set of representative of  $\overline{K}$ . Then the bijection*

$$\begin{aligned} \varphi_{\pi_K}: \mathcal{O}_K &\rightarrow \mathbf{F}_{p^n}[[X]] \\ \sum_{i=0}^{\infty} a_{x_i} \pi_K^i &\mapsto \sum_{i=0}^{\infty} x_i X^i \end{aligned}$$

(which depends on a choice of uniformiser of  $K$ ) induces an isomorphism of ring

$$\overline{\varphi}_{\pi_K}: \mathcal{O}_K/\mathfrak{m}_K^k \rightarrow \mathbf{F}_{p^n}[[X]]/(X^k)$$

*Proof.* Let  $\{a_x\}_{x \in \mathbf{F}_{p^n}}$  be a set of representative of  $\overline{K}$ . Since  $\mathbf{Q}_{p^n} \leq K$ , we can choose the  $a_x$ 's so that they all lie in  $\mathbf{Q}_{p^n}$ . Now, we have  $(a_x + a_y) - a_{x+y} \in (p)$  and  $(a_x a_y) - a_{xy} \in (p)$ . But also, since  $K$  is totally ramified,  $(p) = \mathfrak{m}_K^k$ . Hence, this implies that the map  $\varphi_K$  (which is always a bijection, by the general theory of local fields) is a homomorphism modulo  $\mathfrak{m}_K^k$  and  $(X^k)$ .

To conclude that  $K$  and  $\mathbf{F}_{p^n}((X))$  are at distance  $\frac{1}{2k}$ , it suffices to observe that  $\mathcal{O}_K/\mathfrak{m}_K^{k+1}$  is not isomorphic to  $\mathbf{F}_{p^n}[[X]]/(X^{k+1})$ . But this is clear, since  $p \notin \mathfrak{m}_K^{k+1}$ , hence  $\sum_{i=1}^p 1 \neq 0$  in  $\mathcal{O}_K/\mathfrak{m}_K^{k+1}$ .  $\square$

We need a series of variations on Theorem 5.9, that we now state as corollaries.

**Corollary 5.10.** 1. *Let  $K$  be a totally ramified extension of degree  $k$  of  $\mathbf{Q}_{p^n}$ , and let  $L$  be the unramified quadratic extension of  $K$ . The distance between  $(K, L)$  and  $(\mathbf{F}_{p^n}((X)), \mathbf{F}_{p^{2n}}((X)))$  is  $\frac{1}{2k}$ .*

2. *Let  $K$  be a totally ramified extension of degree  $k$  of  $\mathbf{Q}_{p^n}$ , where  $p$  is an odd prime, and let  $L$  be a ramified quadratic extension of  $K$ . The distance between  $(K, L)$  and  $(\mathbf{F}_{p^n}((X)), \mathbf{F}_{p^n}((\sqrt{X})))$  is  $\frac{1}{2^{2k}}$ .*

3. *Let  $\mathbf{F}_{2^n}((X))[T]/T^2 - \alpha T + X$  be a separable quadratic ramified extension of  $\mathbf{F}_{2^n}((X))$ , with  $\alpha \in (X)$ . Let  $K$  be a totally ramified extension of degree  $k$  of  $\mathbf{Q}_{2^n}$ , and let  $\varphi_{\pi_K}: \mathcal{O}_K \rightarrow \mathbf{F}_{2^n}[[X]]$  be the bijection defined in Theorem 5.9. Finally, let  $a = \varphi_{\pi_K}^{-1}(\alpha) \in \mathcal{O}_K$ . Then  $(K, K[T]/T^2 - aT + \pi_K)$  is  $2k$ -close to  $(\mathbf{F}_{2^n}((X)), \mathbf{F}_{2^n}((X))[T]/T^2 - \alpha T + X)$ .*

*Proof.* 1. As in the proof of Theorem 5.9, let  $\{a_x\}_{x \in \mathbf{F}_{p^n}}$  be a set of representative of  $\overline{K}$  such that  $a_x \in \mathbf{Q}_{p^n}$ , for all  $x \in \mathbf{F}_{p^n}$ .

Since unramified extensions are governed by extensions of the residue field, there exists  $\alpha, \beta \in \mathbf{F}_{p^n}$  such that

$$\begin{aligned} L &\cong K[T]/T^2 - a_{\alpha}T + a_{\beta} \\ \mathbf{F}_{p^{2n}}((X)) &\cong \mathbf{F}_{p^n}((X))[T]/T^2 - \alpha T + \beta \end{aligned}$$

Observing furthermore that

$$\begin{aligned} \mathcal{O}_L/\mathfrak{m}_L^k &\cong \mathcal{O}_K/\mathfrak{m}_K^k \oplus t \cdot \mathcal{O}_K/\mathfrak{m}_K^k \\ \mathbf{F}_{p^{2n}}[[X]]/(X^k) &\cong \mathbf{F}_{p^n}[[X]]/(X^k) \oplus t \cdot \mathbf{F}_{p^n}[[X]]/(X^k) \end{aligned}$$

it is clear (in view of Theorem 5.9) that  $(K, L)$  is  $k$ -close to  $(\mathbf{F}_{p^n}((X)), \mathbf{F}_{p^{2n}}((X)))$ .

To conclude that the distance is  $\frac{1}{2k}$  it suffices to note that if  $(K, L)$  and  $(\mathbf{F}_{p^n}((X)), \mathbf{F}_{p^{2n}}((X)))$  were  $r$ -close for  $r > k$ , then  $K$  and  $\mathbf{F}_{p^n}((X))$  would be  $r$ -close as well, contradicting Theorem 5.9.

- First note that by Lemma 3.3, there exists a uniformiser  $\pi_K \in K$  such that  $L \cong K[T]/T^2 + \pi_K$  (since we avoid by assumption the residue characteristic 2). Also note that for any uniformiser  $\beta \in \mathbf{F}_{p^n}((X))$ , the pair  $(\mathbf{F}_{p^n}((X)), \mathbf{F}_{p^n}((X))[T]/T^2 + \beta)$  is isomorphic to the pair  $(\mathbf{F}_{p^n}((X)), \mathbf{F}_{p^n}((\sqrt{X})))$  (so that despite appearances, there is only one ramified pair on  $\mathbf{F}_{p^n}((X))$ ).

Since

$$\begin{aligned} \mathcal{O}_L/\mathfrak{m}_L^{2k} &\cong \mathcal{O}_K/\mathfrak{m}_K^k \oplus t \cdot \mathcal{O}_K/\mathfrak{m}_K^k \\ \mathbf{F}_{p^n}[\sqrt{X}]/(\sqrt{X}^{2k}) &\cong \mathbf{F}_{p^n}[X]/(X^k) \oplus \sqrt{X} \cdot \mathbf{F}_{p^n}[X]/(X^k) \end{aligned}$$

it is clear (in view of Theorem 5.9) that  $(K, L)$  is  $2k$ -close to  $(\mathbf{F}_{p^n}((X)), \mathbf{F}_{p^n}((\sqrt{X})))$ .

To conclude that the distance is  $\frac{1}{2^k}$  it suffices to note that if  $(K, L)$  and  $(\mathbf{F}_{p^n}((X)), \mathbf{F}_{p^n}((\sqrt{X})))$  were  $r$ -close for  $r > 2k$ , then  $K$  and  $\mathbf{F}_{p^n}((X))$  would be  $\lceil \frac{r}{2} \rceil$ -close as well, contradicting Theorem 5.9.

- The ingredients are similar than for the previous assertions: by Theorem 5.9,  $\mathcal{O}_K/\mathfrak{m}_K^k \cong \mathbf{F}_{2^n}[X]/(X^k)$ . Observing that for a ramified quadratic extension  $L = K[t]$  of  $K$ , we have  $\mathcal{O}_L/\mathfrak{m}_L^{2r} \cong \mathcal{O}_K/\mathfrak{m}_K^r \oplus t \cdot \mathcal{O}_K/\mathfrak{m}_K^r$ , we directly obtain the conclusion. We could also easily conclude that the distance is  $\frac{1}{2^{2k}}$ , but we do not need this information.  $\square$

We also need two further results in the residue characteristic 2 case.

- Lemma 5.11.**
- $(\mathbf{F}_{2^n}((X)), \mathbf{F}_{2^n}((X))[T]/T^2 - X^i T + X)$  is at distance  $\frac{1}{2^{2i}}$  from  $\mathbf{F}_{2^n}((X))$ .
  - Any separable quadratic ramified extension of  $\mathbf{F}_{2^n}((X))$  is of the form  $\mathbf{F}_{2^n}((X))[T]/T^2 - \alpha T + X$ , for some non zero  $\alpha \in (X)$ . Also, given  $i \in \mathbf{N}$ , there are only finitely many extensions (up to isomorphism) of the form  $\mathbf{F}_{2^n}((X))[T]/T^2 - \alpha T + X$  where  $\alpha \in (X^i) \setminus (X^{i+1})$ .

*Proof.* 1. To simplify notations, let  $L = \mathbf{F}_{2^n}((X))[T]/T^2 - X^i T + X$ . Observe that the conjugation action is trivial on  $\mathcal{O}_L/\mathfrak{m}_L^{2i}$ , so that  $\mathcal{O}_L/\mathfrak{m}_L^{2i} \cong \mathbf{F}_{2^n}[X]/(X^i) \oplus \sqrt{X} \cdot \mathbf{F}_{2^n}[X]/(X^i)$ , with trivial conjugation action. Hence,  $(\mathbf{F}_{2^n}((X)), \mathbf{F}_{2^n}((X))[T]/T^2 - X^i T + X)$  is  $2i$ -close from  $\mathbf{F}_{2^n}((X))$ . Now, the conjugation action is non-trivial on  $\mathcal{O}_L/\mathfrak{m}_L^{2i+1}$ , so that the distance is  $\frac{1}{2^{2i+1}}$ .

- By Lemma 3.3, any quadratic ramified extension is of the form  $\mathbf{F}_{2^n}((X))[T]/T^2 - \alpha T + \beta$ , where  $\beta \in (X) \setminus (X^2)$  and  $\alpha \in (X)$ . Now, because  $\mathbf{F}_{2^n}((X))$  has many isomorphisms, such an extension is always (equivariantly) isomorphic to an extension of the desired form. For the last statement, mimicking the proof of [Lan94, Chapter II, §5, Proposition 14], the finiteness follows directly from the compactness of  $(X^i) \setminus (X^{i+1})$ .  $\square$

As in the introduction, let  $\hat{\mathbf{N}}$  denote the one point compactification of  $\mathbf{N}$ .

**Proposition 5.12.** *Let  $p$  be an odd prime number. Then  $\mathcal{L}_{p^n}$  is homeomorphic to  $\hat{\mathbf{N}} \times \{1, 2, 3\}$ . On the other hand,  $\mathcal{L}_{2^n}$  is homeomorphic to  $\hat{\mathbf{N}}^2$ . Furthermore, in  $\mathcal{L}_{2^n}$ , the set of unramified pairs form a clopen subset homeomorphic to  $\hat{\mathbf{N}}$ .*

*Proof.*

**Claim 1.** *Let  $K$  be a local field. If  $|\overline{K}| = p^n$ , then  $K$  is a totally ramified extension of  $\mathbf{Q}_{p^n}$ , or it is isomorphic to  $\mathbf{F}_{p^n}((X))$ .*

*Proof of the claim:* By the classification of local fields,  $K$  is either a finite extension of  $\mathbf{Q}_p$ , or isomorphic to  $\mathbf{F}_{p^n}((X))$  for some prime power  $p^n$ . Since  $\overline{\mathbf{F}_{p^n}((X))} = \mathbf{F}_{p^n}$ , the latter case is clear. For the first case,  $\overline{K} = \mathbf{F}_{p^n}$  if and only if the maximal unramified subextension of  $K$  is  $\mathbf{Q}_{p^n}$ .  $\blacksquare$

**Claim 2.** Let  $(K_k, L_k)$  and  $(K_l, L_l)$  be trivial pairs (respectively unramified, respectively ramified and of residue characteristic not 2). Assume that  $K_k$  and  $K_l$  are totally ramified extension of  $\mathbf{Q}_{p^n}$  such that  $[K_k : \mathbf{Q}_{p^n}] = k < [K_l : \mathbf{Q}_{p^n}] = l$ . Then the distance between  $(K_k, L_k)$  and  $(K_l, L_l)$  is  $\frac{1}{2^k}$ .

*Proof of the claim:* We observed in Lemma 5.7 that  $\mathcal{L}$  is a non-archimedean metric space, and hence every triangle is isosceles. Thus, the distance between  $(K_k, L_k)$  and  $(K_l, L_l)$  is either  $\frac{1}{2^k}$  or  $\frac{1}{2^l}$  (taking in each case as a comparison point the corresponding pair in positive characteristic, and using Theorem 5.9, or Corollary 5.10). But in the latter case, since being  $l$ -close is an equivalence relation, we would conclude that  $(K_k, L_k)$  is  $l$ -close to  $\mathbf{F}_{p^n}((X))$ , which would contradict Theorem 5.9 or Corollary 5.10. ■

**Claim 3.** There are only finitely many totally ramified extension of degree  $\leq k$  of a local field of characteristic 0.

*Proof of the claim:* This is just a well-known corollary of the so called Krasner's Lemma. A proof of Claim 3 can be found in [Lan94, Chapter II, §5, Proposition 14]. ■

**Claim 4.** Let  $(K, L) \in \mathcal{L}_{p^n}$ . If  $K$  is of characteristic 0, the pair  $(K, L)$  is isolated in  $\mathcal{L}_{p^n}$ .

*Proof of the claim:* Since unramified pairs are at distance 1 from other kind of pairs, it follows from Claim 2 and Claim 3 that unramified pairs of characteristic 0 are isolated in  $\mathcal{L}_{p^n}$ .

When  $p$  is an odd prime, ramified pairs are at distance  $\frac{1}{2}$  from trivial pairs, and there are only 2 different quadratic ramified extension of a given local field (since  $p$  is odd) hence the result follows again from Claim 2 and Claim 3 in this case.

Finally, when  $p = 2$ , let  $(K, L)$  be a trivial or ramified pair of characteristic 0 belonging to  $\mathcal{L}_{2^n}$ . By definition, if  $(K, L)$  is  $r$ -close to  $(\tilde{K}, \tilde{L})$ , then  $K$  is  $\lceil \frac{r}{2} \rceil$ -close to  $\tilde{K}$ . Hence, by Claim 2 for trivial pairs,  $(K, L)$  is isolated from pairs  $(\tilde{K}, \tilde{L})$  where  $\tilde{K}$  is either of characteristic 2, or  $[\tilde{K} : \mathbf{Q}_{2^n}] \neq [K : \mathbf{Q}_{2^n}]$ . But there are only finitely many pairs  $(K, L)$  with  $[K : \mathbf{Q}_{2^n}] = k$  by Claim 3. Hence, the conclusion. ■

**Claim 5.**  $\mathcal{L}_{p^n}$  is a countable space.

*Proof of the claim:* By Claim 3, there are only countably many pairs of characteristic 0. For pairs of positive characteristic, if  $p$  is odd, there is only one pair of each type (recall that we consider pairs up to isomorphism). If  $p = 2$ , there is one trivial pair and one unramified pair, and there are countably many ramified pairs of characteristic 2 by Lemma 5.11. ■

We are now able to deduce the homeomorphism type of  $\mathcal{L}_{p^n}$ : for  $p$  any prime, the unramified pairs are isolated from other kind of pairs in  $\mathcal{L}_{p^n}$ . Furthermore, unramified pairs of characteristic 0 are isolated by Claim 4, and the unramified pair of positive characteristic is an accumulation point by Corollary 5.10. Hence, by [MS20, Théorème 1], unramified pairs account for one disjoint copy of  $\hat{\mathbf{N}}$ .

When  $p$  is odd, trivial pairs (respectively ramified pairs) are isolated from ramified pairs (respectively trivial pairs), the characteristic 0 ones are isolated by Claim 4, and the unique pair of positive characteristic is an accumulation points by Theorem 5.9 and Corollary 5.10, so that we obtain two more disjoint copies of  $\hat{\mathbf{N}}$ .

Finally, when  $p = 2$ , since pairs of characteristic 0 are isolated by Claim 4, the first Cantor Bendixson derivative  $\mathcal{L}_{2^n}^{(1)}$  contains only pairs of positive characteristic, and  $\mathcal{L}_{2^n}^{(1)}$  contains all of them by Corollary 5.10. Also, by Lemma 5.11, ramified pairs are isolated in  $\mathcal{L}_{2^n}^{(1)}$ , and the trivial pair  $\mathbf{F}_{2^n}((X))$  is an accumulation point in  $\mathcal{L}_{2^n}^{(1)}$ . So that again by [MS20, Théorème 1], we get the result. □

## 5.2 The topological space of division algebras

In Section 5.1, we studied convergence in the space  $\mathcal{L}$  of pairs of local fields. This subsequently allows us to conclude convergence in a corresponding Chabauty space (see Theorem 6.5), in the case of quasi-split (absolutely simple, simply connected) algebraic groups of rank 1 (i.e. in the  $\mathrm{SL}_2(K)$  case and the  $\mathrm{SU}_3$  case). It turns out that groups of the form  $\mathrm{SL}_2(D)$  with  $[D : K] > 1$  do not converge to quasi-split groups in the Chabauty space, and hence we can treat arithmetical convergence for division algebras separately from arithmetical convergences of pairs of local fields.

**Definition 5.13.** Let  $\mathcal{D}$  be the set of finite dimensional division algebras  $D$  over a local field  $K$ , up to isomorphism. Let also  $\mathcal{D}_{p^n} = \{D \in \mathcal{D} \mid |\overline{D}| = p^n\}$ . As in Section 5.1, we say that  $D_1$  is  $r$ -close to  $D_2$  if and only if there exists an isomorphism  $\mathcal{O}_{D_1}/\mathfrak{m}_{D_1}^r \rightarrow \mathcal{O}_{D_2}/\mathfrak{m}_{D_2}^r$ .

Again, this notion of closeness induces a non-archimedean metric on  $\mathcal{D}$ , by defining

$$d: \mathcal{D} \times \mathcal{D} \rightarrow \mathbf{R}_{\geq 0}: d(D_1, D_2) = \inf\left\{\frac{1}{2^r} \mid D_1 \text{ is } r\text{-close to } D_2\right\}$$

It is then quite straightforward to work out the homeomorphism type of  $\mathcal{D}_{p^n}$ .

**Proposition 5.14.** Let  $p$  be a prime number. Then  $\mathcal{D}_{p^n}$  is homeomorphic to  $\hat{\mathbf{N}} \times \{1, \dots, n\}$ .

*Proof.* Let  $D \in \mathcal{D}_{p^n}$ . Then  $D$  is isomorphic to the cyclic algebra  $(E/K, \sigma^r, \pi_K)$  (see Definition B.1), where  $[E : K] = d$  divides  $n$ ,  $r \in (\mathbf{Z}/d\mathbf{Z})^\times$  and  $|\overline{K}| = p^{\frac{n}{d}}$ . Furthermore, it is easily seen that if  $D_1 = (E_1/K_1, \sigma^{r_1}, \pi_{K_1})$  is 2-close to  $D_2 = (E_2/K_2, \sigma^{r_2}, \pi_{K_2})$ , then  $[E_1 : K_1] = [E_2 : K_2]$  and  $r_1 = r_2$ .

Using Theorem 5.9 and the explicit description of central division algebra given in Appendix B, we see that a point in  $\mathcal{D}_{p^n}$  is isolated if and only if the corresponding division algebra is of characteristic 0 (see also [dlST15, Theorem 1.2]). Finally,  $\mathcal{D}_{p^n}$  is a countable space, and it is readily seen that the number of positive characteristic division algebras in  $\mathcal{D}_{p^n}$  is equal to  $\sum_{d|n} |(\mathbf{Z}/d\mathbf{Z})^\times| = n$ .

Hence, the result follows from [MS20, Théorème 1].  $\square$

## 6 Continuity from local fields to subgroups of $\mathrm{Aut}(T)$

**Definition 6.1.** Let  $(K, L) \in \mathcal{L}$ .

1. If  $(K, L)$  is trivial, we associate to it the group  $\mathrm{SL}_2(K)$ .
2. if  $(K, L)$  is ramified or unramified, we associate to it the group  $\mathrm{SU}_3^{L/K}(K)$ .

The associated group is denoted  $G_{(K,L)}$ . Similarly, we associate to  $D \in \mathcal{D}$  the group  $G_D = \mathrm{SL}_2(D)$  (note that if  $D = K$ , the two definitions coincide).

**Proposition 6.2.** Let  $(K_1, L_1)$  and  $(K_2, L_2)$  be two elements in  $\mathcal{L}$  that are  $r$ -close, with  $r > 1$ . Let  $G_i$  be the algebraic group associated with  $(K_i, L_i)$ . Then  $(P_0^{0,r})_{G_1} \cong (P_0^{0,r})_{G_2}$ , and  $\mathcal{I}_{G_1}^{0,r}$  is equivariantly in bijection with  $\mathcal{I}_{G_2}^{0,r}$ , except when  $(K_1, L_1)$  is a ramified pair and  $(K_2, L_2)$  is trivial. In this latter case,  $(P_0^{0,r-1})_{G_1} \cong (P_0^{0,r-1})_{G_2}$ , and  $\mathcal{I}_{G_1}^{0,r-1}$  is equivariantly in bijection with  $\mathcal{I}_{G_2}^{0,r-1}$ .

*Proof.* We prove it on a case by case analysis.

1. When the pair are both trivial, the isomorphism  $\mathcal{O}_{K_1}/\mathfrak{m}_{K_1}^r \cong \mathcal{O}_{K_2}/\mathfrak{m}_{K_2}^r$  induces an isomorphism  $\varphi: (P_0^{0,r})_{G_1} = \mathrm{SL}_2(\mathcal{O}_{K_1}/\mathfrak{m}_{K_1}^r) \cong \mathrm{SL}_2(\mathcal{O}_{K_2}/\mathfrak{m}_{K_2}^r) = (P_0^{0,r})_{G_2}$ . Define a linear map  $f: \mathbf{R} \rightarrow \mathbf{R}: x \mapsto x \frac{\omega(\pi_{K_2})}{\omega(\pi_{K_1})}$ . It is clear that for all  $x \in [-\omega(\pi_{K_1}^r); \omega(\pi_{K_1}^r)]$ ,  $\varphi$  restricts to an isomorphism  $(P_x^{0,r})_{G_1} \cong (P_{f(x)}^{0,r})_{G_2}$ . Furthermore,

$$\varphi(T^{0,r})_{G_1} = (T^{0,r})_{G_2}$$

$$\varphi(M^{0,r})_{G_1} = (M^{0,r})_{G_2}$$

and for all  $n \in N^{0,r}$ ,  $f(n.x) = \varphi(n).f(x)$ . Hence, the map  $\mathcal{I}_{G_1}^{0,r} \rightarrow \mathcal{I}_{G_2}^{0,r}: [(g,x)]^{0,r} \mapsto [(\varphi(g); f(x))]^{0,r}$  is a  $\varphi$ -equivariant bijection.

2. When the pair are both ramified or both unramified, the argument is the same than for the previous case: the conjugation equivariant isomorphism  $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r \cong \mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^r$  induces an isomorphism  $\varphi$

$$\begin{array}{ccc} \mathrm{SL}_3(\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r) & \cong & \mathrm{SL}_3(\mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^r) \\ \downarrow & & \downarrow \\ (P_0^{0,r})_{G_1} & \xrightarrow{\varphi} & (P_0^{0,r})_{G_2} \end{array}$$

Define a linear map  $f: \mathbf{R} \rightarrow \mathbf{R}: x \mapsto x \frac{\omega(\pi_{L_2})}{\omega(\pi_{L_1})}$ . It is clear that for all  $x \in [-\omega(\pi_{L_1}^r); \omega(\pi_{L_1}^r)]$ ,  $\varphi$  restricts to an isomorphism  $(P_x^{0,r})_{G_1} \cong (P_{f(x)}^{0,r})_{G_2}$ . Furthermore,

$$\begin{aligned} \varphi(T^{0,r})_{G_1} &= (T^{0,r})_{G_2} \\ \varphi(M^{0,r})_{G_1} &= (M^{0,r})_{G_2} \end{aligned}$$

and for all  $n \in N^{0,r}$ ,  $f(n.x) = \varphi(n).f(x)$ . Hence, the map  $\mathcal{I}_{G_1}^{0,r} \rightarrow \mathcal{I}_{G_2}^{0,r}: [(g,x)]^{0,r} \mapsto [(\varphi(g); f(x))]^{0,r}$  is a  $\varphi$ -equivariant bijection.

3. Recall that unramified pairs are isolated from pairs of other types, and that ramified pairs in residue characteristic not 2 are at distance  $\frac{1}{2}$  from trivial pairs. Since we assume that  $r > 1$ , there just remains to examine the case when a trivial pair is  $r$ -close to a ramified pair in residue characteristic 2.

Without loss of generality,  $(K_1, L_1)$  is the ramified pair. Let  $t \in L_1$  be such that  $t^2 = \alpha t - \beta$ , where  $t, \alpha$  and  $\beta$  are as in Lemma 3.3. Since  $(K_1, L_1)$  is  $r$ -close to  $(K_2, L_2)$  and  $(K_2, L_2)$  is a trivial pair, in particular the conjugation is trivial modulo  $\mathfrak{m}_{L_1}^r$ . Hence, if  $\alpha \neq 0$  (respectively if  $\alpha = 0$ ),  $\omega(2) \geq \omega(\alpha) = \omega(\pi_K^{i_0}) \geq \omega(\pi_L^r)$  (respectively  $\omega(2) = \omega(\pi_K^{i_0}) \geq \omega(\pi_L^{r-1})$ ), so that we have  $r - 1 \leq 2i_0$ , as needed to apply Definition 4.6 to  $r - 1$ .

That being said, we can proceed as for the other cases: the isomorphism  $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^{r-1} \cong \mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^{r-1}$  induces an isomorphism  $\varphi: (P_0^{0,r-1})_{G_1} = \mathrm{SL}_2(\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^{r-1}) \cong \mathrm{SL}_2(\mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^{r-1}) = (P_0^{0,r-1})_{G_2}$ . Define a linear map  $f: \mathbf{R} \rightarrow \mathbf{R}: x \mapsto x \frac{\omega(\pi_{L_2})}{\omega(\pi_{L_1})}$ . It is clear that for all  $x \in [-\omega(\pi_{L_1}^{r-1}); \omega(\pi_{L_1}^{r-1})]$ ,  $\varphi$  restricts to an isomorphism  $(P_x^{0,r-1})_{G_1} \cong (P_{f(x)}^{0,r-1})_{G_2}$ . Furthermore,

$$\begin{aligned} \varphi(T^{0,r-1})_{G_1} &= (T^{0,r-1})_{G_2} \\ \varphi(M^{0,r-1})_{G_1} &= (M^{0,r-1})_{G_2} \end{aligned}$$

and for all  $n \in N^{0,r-1}$ ,  $f(n.x) = \varphi(n).f(x)$ . Hence, the map  $\mathcal{I}_{G_1}^{0,r-1} \rightarrow \mathcal{I}_{G_2}^{0,r-1}: [(g,x)]^{0,r} \mapsto [(\varphi(g); f(x))]^{0,r}$  is a  $\varphi$ -equivariant bijection.  $\square$

**Proposition 6.3.** *Let  $D_1$  and  $D_2$  be two elements in  $\mathcal{D}$  that are  $rd_1$ -close, with  $r \geq 1$  and with  $d_1 = \sqrt{[D_1 : K_1]}$ . We have  $\sqrt{[D_1 : K_1]} = \sqrt{[D_2 : K_2]} = d$ . Let  $G_i$  be the algebraic group associated with  $D_i$ . Then  $(P_0^{0,rd})_{G_1} \cong (P_0^{0,rd})_{G_2}$ , and  $\mathcal{I}_{G_1}^{0,rd}$  is equivariantly in bijection with  $\mathcal{I}_{G_2}^{0,rd}$ .*

*Proof.* The proof is the same than the proof of Proposition 6.2. The isomorphism  $\mathcal{O}_{D_1}/\mathfrak{m}_{D_1}^{rd} \cong \mathcal{O}_{D_2}/\mathfrak{m}_{D_2}^{rd}$  induces an isomorphism  $\varphi: (P_0^{0,rd})_{G_1} = \mathrm{SL}_2(\mathcal{O}_{D_1}/\mathfrak{m}_{D_1}^{rd}) \cong \mathrm{SL}_2(\mathcal{O}_{D_2}/\mathfrak{m}_{D_2}^{rd}) = (P_0^{0,rd})_{G_2}$ . Define a linear map  $f: \mathbf{R} \rightarrow \mathbf{R}: x \mapsto x \frac{\omega(\pi_{D_2})}{\omega(\pi_{D_1})}$ . It is clear that for all  $x \in [-\omega(\pi_{D_1}^{rd}); \omega(\pi_{D_1}^{rd})]$ ,  $\varphi$  restricts to an isomorphism  $(P_x^{0,rd})_{G_1} \cong (P_{f(x)}^{0,rd})_{G_2}$ . Furthermore,

$$\varphi(T^{0,rd})_{G_1} = (T^{0,rd})_{G_2}$$

$$\varphi(M^{0,rd})_{G_1} = (M^{0,rd})_{G_2}$$

and for all  $n \in N^{0,rd}$ ,  $f(n.x) = \varphi(n).f(x)$ . Hence, the map  $\mathcal{I}_{G_1}^{0,rd} \rightarrow \mathcal{I}_{G_2}^{0,rd}: [(g, x)]^{0,rd} \mapsto [(\varphi(g); f(x))]^{0,rd}$  is a  $\varphi$ -equivariant bijection.  $\square$

We can finally go back to our original problem, which is to study convergence of algebraic groups in the Chabauty space of  $\text{Aut}(T)$ . We first discuss the homomorphism  $G \rightarrow \text{Aut}(T)$  (for  $G$  equal to  $\text{SL}_2(D)$  or  $\text{SU}_3^{L/K}(K)$ ).

**Proposition 6.4.** *Let  $G$  be either  $\text{SL}_2(D)$  or  $\text{SU}_3(K)$ , and let  $T_G$  be its associated Bruhat–Tits tree (Definition 3.10). The induced homomorphism  $\hat{\cdot}: G \rightarrow \text{Aut}(T_G)$  is continuous with closed image, and the kernel is equal to the center of  $G$ .*

*Proof.* In each case, the group  $P_x$  is really the stabiliser of  $[(\text{Id}, x)] \in \mathcal{I}$  (see Remark 3.11). Since a basic identity neighbourhood in  $\text{Aut}(T)$  is given by intersecting finitely many vertices stabilisers, the continuity follows. The fact that the image is closed follows from the general argument in [BM96, Lemma 5.3]. Finally, the kernel can also be seen directly from the explicit description of  $P_x$ . Indeed, if  $g$  is in the intersection  $\bigcap_{x \in \mathbf{R}} P_x$ , then  $g$  is diagonal. But also, the conjugation action of  $g$  on root groups needs to be trivial, so that  $g$  is in the center of  $G$ . Conversely, the center of  $G$  clearly acts trivially on  $\mathcal{I}$ , which concludes the proof.  $\square$

The convergence is then a more or less direct consequence of Theorem 4.32.

**Theorem 6.5.** *Let  $((K_i, L_i))_{i \in \mathbf{N}}$  be a sequence in  $\mathcal{L}$  which converges to  $(K, L)$ . Let  $T = T_{(K, L)}$  (respectively  $T_i = T_{(K_i, L_i)}$ ) be the Bruhat–Tits tree of  $G = G_{(K, L)}$  (respectively  $G_i = G_{(K_i, L_i)}$ ). For  $N$  big enough and for all  $i \geq N$ , there exist isomorphisms  $T_i \cong T$  such that the induced embeddings  $\hat{G}_i \hookrightarrow \text{Aut}(T)$  make  $(\hat{G}_i)_{i \geq N}$  converge to  $\hat{G}$  in the Chabauty topology of  $\text{Aut}(T)$ .*

**Remark 6.6.** The convergence depends on a choice of specific isomorphisms  $T_i \cong T$ , or in other words it depends on choosing how  $\hat{G}_i$  sits in  $\text{Aut}(T)$ . This dependence is not problematic since for two isomorphic closed subgroups  $H, H'$  of  $\text{Aut}(T)$  both acting 2-transitively on  $\partial T$ , there exists  $g$  in the fixator of  $e_0$  such that  $gHg^{-1} = H'$ , where  $e_0$  is any edge of  $T$  (see [Rad15, Proposition A.1], and recall also that  $H$  acts transitively on the edges of  $T$ ). Hence, for other choices of embeddings, the sequence converges to a conjugate of  $\hat{G}$  in  $\text{Aut}(T)$ . Recall also that we introduced the space  $\mathcal{S}_T$  in the introduction precisely to avoid this dependence.

The main step of the proof is to establish that the sequence of stabilisers  $((\hat{P}_0)_{G_i})_{i \geq N}$  converges to the stabiliser  $(\hat{P}_0)_G$  in  $\text{Aut}(T)$ . From there, we can conclude that  $(\hat{G}_i)_{i \geq N}$  converges to  $\hat{G}$  from general theory.

*Proof.* As we recall in the introduction, the Bruhat–Tits tree  $T$  is regular of degree  $p^n + 1$  (respectively semiregular of degree  $(p^{3n} + 1; p^n + 1)$ ) if the pair  $(K, L)$  is trivial or ramified (respectively unramified) and belongs to  $\mathcal{L}_{p^n}$ . This shows that there exists  $N$  such that for all  $i \geq N$ ,  $T_i \cong T$ .

Passing to a subsequence, we can assume that  $(K_i, L_i)$  is  $(2i+1)$ -close to  $(K, L)$ . We now define an explicit isomorphism  $f_i: T_i = \mathcal{I}_{G_i} \rightarrow \mathcal{I}_G = T$  as follows: let  $\mathcal{I}_{G_i}^{0,2i} \cong \mathcal{I}_G^{0,2i}$  be the isomorphism given by Proposition 6.2. By Theorem 4.32, this gives an isomorphism on balls of radius  $2i$ :  $\mathcal{I}_{G_i} \supset B_0(2i) \cong B_0(2i) \subset \mathcal{I}_G$  (recall that by Lemma 4.31,  $B_0(2i)$  is really the ball of radius  $2i$  on the tree  $\mathcal{I}_G$ ). As  $\mathcal{I}_{G_i}$  is a semiregular tree of the same bidegree than  $\mathcal{I}_G$ , we can extend this isomorphism of balls to an isomorphism  $f_i: \mathcal{I}_{G_i} \rightarrow \mathcal{I}_G$  (this extension is of course not unique, but we choose one such). By means of  $f_i$ , we get an embedding  $\hat{G}_i \rightarrow \text{Aut}(T)$ .

We claim that  $((\hat{P}_0)_{G_i})_{i \in \mathbf{N}}$  converges to  $(\hat{P}_0)_G$ . According to [CR16, Lemma 2.1], there are two things to verify.

1. Let  $(\hat{h}_i)$  be a sequence such that  $\hat{h}_i \in (\hat{P}_0)_{G_i}$ , and assume that  $\hat{h}_i$  converges to  $\hat{h}$  in  $\text{Aut}(T)$ . We have to show that  $\hat{h} \in (\hat{P}_0)_G$ . For all  $i$ , let  $h_i \in (P_0)_{G_i}$  be an inverse image of  $\hat{h}_i$  under  $\hat{\cdot}: G_i \rightarrow \text{Aut}(T)$ . Let  $\bar{h}_i = p_{2i}(h_i) \in (P_0^{0,2i})_{G_i}$ . Let  $\varphi_{2i}: (P_0^{0,2i})_{G_i} \cong (P_0^{0,2i})_G$  be the



isomorphism given in Proposition 6.2. By Corollary 4.26, there exists  $\tilde{h}_i \in (P_0)_G$  which is an inverse image of  $\varphi_{2i}(\tilde{h}_i)$  under  $p_{2i}: (P_0)_G \rightarrow (P_0^{0,2i})_G$ . Now, because all the identifications were equivariant, the action of  $\tilde{h}_i$  on the ball of radius  $2i$  around 0 is the same than the action of  $\hat{h}_i$  on this ball. Hence,  $(\tilde{h}_i)$  converges to  $\hat{h}$  as well. But  $(\hat{P}_0)_G$  is a closed subgroup of  $\text{Aut}(T)$  (by Proposition 6.4), hence  $\hat{h} \in (\hat{P}_0)_G$ , as wanted.

2. Conversely, given an element  $\hat{h} \in (\hat{P}_0)_G$ , we have to find a sequence  $(\hat{h}_i)$  of elements in  $(\hat{P}_0)_{G_i}$  such that  $\hat{h}_i$  converges to  $\hat{h} \in \text{Aut}(T)$ . It suffices to follow the path of identifications in reverse : let  $h$  be an inverse image of  $\hat{h}$  under  $\hat{\cdot}: G \rightarrow \text{Aut}(T)$ . Let  $\bar{h}_i = p_{2i}(h) \in (P_0^{0,2i})_G$ , and let  $\varphi_{2i}: (P_0^{0,2i})_G \cong (P_0^{0,2i})_{G_i}$  be the isomorphism given in Proposition 6.2. For all  $i$ , let  $h_i$  be an inverse image of  $\varphi_{2i}(\bar{h}_i)$  under  $p_{2i}: (P_0)_{G_i} \rightarrow (P_0^{0,2i})_{G_i}$ , which exists by Corollary 4.26. Now, because all the identifications were equivariant, the action of  $h_i$  on the ball of radius  $i$  around 0 is the same than the action of  $h$  on this ball. Hence,  $(h_i)$  converges to  $\hat{h}$ , as wanted.

Finally, from the convergence of  $((\hat{P}_0)_{G_i})_{i \geq N}$  to  $(\hat{P}_0)_G$ , we can formally deduce the convergence of  $(\hat{G}_i)_{i \geq N}$  to  $\hat{G}$ . Indeed,  $(\hat{G}_i)_{i \geq N}$  subconverges to a topologically simple group  $H$ , by [CR16, Theorem 1.2]. But since  $((\hat{P}_0)_{G_i})_{i \geq N}$  converges to  $(\hat{P}_0)_G$ ,  $H$  has an open compact subgroup isomorphic to  $(\hat{P}_0)_G$ . Hence, by [CS15, Corollary 1.3],  $H$  is algebraic. And hence, by [Pin98, Corollary 0.3],  $H \cong G$ . Since by the same argument, any subsequence of  $(\hat{G}_i)_{i \geq N}$  subconverges to  $\hat{G}$ , we conclude that  $(\hat{G}_i)_{i \geq N}$  converges to  $\hat{G}$ .  $\square$

Similarly, we can prove the corresponding statement for sequences in  $\mathcal{D}$ .

**Theorem 6.7.** *Let  $(D_i)_{i \in \mathbb{N}}$  be a sequence in  $\mathcal{D}$  which converges to  $D$ . Let  $T = T_D$  (respectively  $T_i = T_{D_i}$ ) be the Bruhat–Tits tree of  $G = G_D$  (respectively  $G_i = G_{D_i}$ ). For  $N$  big enough and for all  $i \geq N$ , there exist isomorphisms  $T_i \cong T$  such that the induced embeddings  $\hat{G}_i \hookrightarrow \text{Aut}(T)$  make  $(\hat{G}_i)_{i \geq N}$  converge to  $\hat{G}$  in the Chabauty topology of  $\text{Aut}(T)$ .*

*Proof.* The Bruhat–Tits tree  $T_{D_i}$  is the regular tree of degree  $p^n + 1$  if and only if  $D_i$  belongs to  $\mathcal{D}_{p^n}$ . Hence there exists  $N$  such that for all  $i \geq N$ ,  $T_i \cong T$ .

Passing to a subsequence, we can assume that  $D_i$  is  $(di)$ -close to  $D$ , where  $D$  is of degree  $d$  over its center. Hence, for  $i \geq 1$ ,  $D_i$  is also of degree  $d$  over its center. We now define an explicit isomorphism  $f_i: T_i = \mathcal{I}_{G_i} \rightarrow \mathcal{I}_G = T$  as follows: let  $\mathcal{I}_{G_i}^{0,di} \cong \mathcal{I}_G^{0,di}$  be the isomorphism given by Proposition 6.3. By Theorem 4.32, this gives an isomorphism on balls of radius  $di$ :  $\mathcal{I}_{G_i} \supset B_0(di) \cong B_0(di) \subset \mathcal{I}_G$  (recall that by Lemma 4.31,  $B_0(di)$  is really the ball of radius  $di$  on the tree  $\mathcal{I}_G$ ). As  $\mathcal{I}_{G_i}$  is a regular tree of the same degree than  $\mathcal{I}_G$ , we can extend this isomorphism of balls to an isomorphism  $f_i: \mathcal{I}_{G_i} \rightarrow \mathcal{I}_G$  (this extension is of course not unique, but we choose one such). By means of  $f_i$ , we get an embedding  $\hat{G}_i \hookrightarrow \text{Aut}(T)$ .

Now, the end of the proof is word for word the same than the corresponding end of the proof of Theorem 6.5, upon replacing all 2's with  $d$ 's, and upon replacing the reference to Proposition 6.2 with a reference to Proposition 6.3.  $\square$

*Proof of Theorem 1.3.* Let  $T$  be a semiregular tree and let  $\mathcal{L}_T = \{(K, L) \in \mathcal{L} \mid \text{the Bruhat–Tits tree of } G_{(K,L)} \text{ is isomorphic to } T\}$ . By Remark 3.12 and Proposition 5.12,  $\mathcal{L}_T$  is a compact space. Now, by Theorem 6.5, the map  $\mathcal{L}_T \rightarrow \mathcal{S}_T: (K, L) \mapsto \hat{G}_{(K,L)}$  is continuous. We claim that it is injective as well. Indeed, if  $\hat{G}_{(K_1,L_1)}$  is abstractly isomorphic to  $\hat{G}_{(K_2,L_2)}$ , then by [BT73, Corollaire 8.13], the corresponding adjoint algebraic groups  $\text{Ad } \mathbf{G}_1$  and  $\text{Ad } \mathbf{G}_2$  are algebraically isomorphic over an isomorphism of fields  $K_1 \cong K_2$ . Since  $\text{Ad } \mathbf{G}_1$  (respectively  $\text{Ad } \mathbf{G}_2$ ) is quasi-split, there exists a smallest extension splitting it ([BT84a, 4.1.2]), namely  $L_1$  (respectively  $L_2$ ). Hence,  $(K_1, L_1) \cong (K_2, L_2)$ , as wanted.

To summarise,  $\mathcal{L}_T \rightarrow \mathcal{S}_T: (K, L) \mapsto \hat{G}_{(K,L)}$  is an injective continuous map whose source is a compact space, hence it is a homeomorphism onto its image. Now, the explicit description given in Theorem 1.3 follows from Remark 3.12 and Proposition 5.12.  $\square$



*Proof of Theorem 1.6.* Let  $T$  be a regular tree and let  $\mathcal{D}_T = \{D \in \mathcal{D} \mid \text{the Bruhat–Tits tree of } G_D \text{ is isomorphic to } T\}$ . By Remark 3.12 and Proposition 5.14,  $\mathcal{D}_T$  is a compact space. Now, by Theorem 6.7, the map  $\mathcal{D}_T \rightarrow \mathcal{S}_T: D \mapsto \hat{G}_D$  is continuous. Let  $D_1$  and  $D_2$  be central division algebras over  $K_1$  and  $K_2$  respectively, with respective degree  $d_1, d_2$  and Hasse invariant  $r_1, r_2$  (as defined in Definition B.2). We claim that  $\hat{G}_{D_1} = \hat{G}_{D_2}$  if and only if  $K_1 \simeq K_2$ ,  $d_1 = d_2$  and  $r_1 = \pm r_2$ . Indeed, if  $\hat{G}_{D_1}$  is abstractly isomorphic to  $\hat{G}_{D_2}$ , then by [BT73, Corollaire 8.13], the corresponding adjoint algebraic groups  $\text{Ad } \mathbf{G}_1$  and  $\text{Ad } \mathbf{G}_2$  are algebraically isomorphic over an isomorphism of fields  $K_1 \cong K_2$ . Now, according to [KMRT98, Remark 26.11], this is only possible if  $D_1 \cong D_2$  or  $D_1 \cong D_2^{\text{opp}}$ , which is equivalent to the given condition.

To summarise, let  $\mathcal{D}_T / \sim_{\text{opp}}$  be the space  $\mathcal{D}_T$  modulo the equivalence relation  $D_1 \sim_{\text{opp}} D_2$  if and only if  $D_1 \cong D_2$  or  $D_1 \cong D_2^{\text{opp}}$ . We proved that  $\mathcal{D}_T / \sim_{\text{opp}} \rightarrow \mathcal{S}_T: D \mapsto \hat{G}_D$  is an injective continuous map whose source is a compact space, hence it is a homeomorphism onto its image. Now, the explicit description given in Theorem 1.6 follows from Remark 3.12 and Proposition 5.14.

To be able to conclude that for  $T$  the  $(p^n + 1)$ -regular tree,  $\mathcal{S}_T^{\text{SL}_2(D)}$  is homeomorphic to  $\hat{\mathbf{N}} \times \{1, \dots, \lceil \frac{n+1}{2} \rceil\}$ , one has to count the number of division algebras in  $\mathcal{D}_T / \sim_{\text{opp}}$  of characteristic  $p$ . But there is only one such division algebra in  $\mathcal{D}_T / \sim_{\text{opp}}$  of degree 1 over its center, one such division algebra in  $\mathcal{D}_T / \sim_{\text{opp}}$  of degree 2 over its center if 2 divides  $n$ , and for all  $3 \leq d$  dividing  $n$ , there are  $\frac{\varphi(d)}{2}$  such division algebras in  $\mathcal{D}_T / \sim_{\text{opp}}$  of degree  $d$  over their center (where  $\varphi$  denotes Euler's totient function). Hence, if  $n$  is even (respectively odd), we have  $2 + \sum_{d|n, d \geq 3} \frac{\varphi(d)}{2}$  (respectively  $1 + \sum_{d|n, d \geq 3} \frac{\varphi(d)}{2}$ ) division algebras of characteristic  $p$  in  $\mathcal{D}_T / \sim_{\text{opp}}$ . Using that  $\sum_{d|n} \varphi(d) = n$ , we readily get the conclusion.  $\square$

## A Comparison with the original Bruhat–Tits definitions

The purpose of this appendix is to show that our definition of the Bruhat–Tits tree agrees with the one in [BT72, 7.4.1 and 7.4.2]. Since the relative rank of  $\text{SL}_2(D)$  and  $\text{SU}_3$  is 1, it is already clear that the apartment  $A$  is indeed isomorphic to  $\mathbf{R}$ . The main task is to show that our group  $P_x$  is the same as the group  $\hat{P}_x$  used to define the equivalence relation in [BT72, 7.4.1].

In the  $\text{SL}_2(D)$  case, the explicit description of  $\hat{P}_x$  is given in [BT72, Corollaire 10.2.9], that we take as a definition.

**Definition A.1** ([BT72, Corollaire 10.2.9]). Let  $\{a_1, a_2\}$  be the canonical basis of  $\mathbf{R}^2$ , and let  $a_{ij} = a_j - a_i$  ( $i, j \in \{1, 2\}$ ). We can see  $\mathbf{R}$  as a vector space  $V$ , dual of the vector space  $V^* = \mathbf{R} \cdot a_{12}$ . Now, for  $x \in \mathbf{R}$ ,  $\hat{P}_x = \{g \in \text{SL}_2(K) \mid \omega(g_{ij}) \geq a_{ji}(x), \text{ for all } 1 \leq i, j \leq 2\}$ .

Note that we can omit the factor  $(r+1)^{-1}\delta$  appearing in loc. cit. since by definition,  $\delta = \omega(\det(g)) = \omega(1) = 0$ .

This description obviously depends on the identification of  $\mathbf{R}$  as the dual of  $V^*$ . Now, if we furthermore impose the condition  $a_{12} = \text{Id}: \mathbf{R} \rightarrow \mathbf{R}$ , then  $\hat{P}_x$  is indeed equal to the group  $P_x$  of Definition 3.2. To end the comparison between [BT72, Définition 7.4.2] and our Definition 3.10, one has to show that the maps  $\nu: N \rightarrow \text{Aff}(\mathbf{R})$  are the same. This is easily obtained by comparing [BT72, Proposition 10.2.5 (ii)] with our Definition 3.9.

In the  $\text{SU}_3$  case, as in Definition 2.1, we index the rows and the columns of a 3-by-3 matrix by  $\{-1, 0, 1\}$ . Let  $a_1$  be a non-trivial element of  $\mathbf{R}^*$ , and set  $a_{-1} = -a_1$  and  $a_0 = 0$ . We now take some time to spell out the definition of  $\bar{\omega}_{ij}$  as defined in [BT72, 10.1.27].

**Definition A.2.** Recall the definition of the element  $l \in L$  we introduced in Definition 3.5. Namely,  $l = \begin{cases} t\alpha^{-1} & \text{if } \alpha \neq 0 \\ \frac{1}{2} & \text{if } \alpha = 0 \end{cases}$ , where  $t$  and  $\alpha$  are as in Lemma 3.3.

**Lemma A.3.** Let  $L^1 = \{x \in L \mid x + \bar{x} = 1\}$  and  $L_{\max}^1 = \{x \in L^1 \mid \omega(x) = \sup\{\omega(x) \mid x \in L^1\}\}$ . The element  $l \in L$  in Definition A.2 belongs to  $L_{\max}^1$ .

*Proof.* See [BT84a, 4.3.3 (ii)].  $\square$

**Definition A.4** ([BT72, 10.1.20]). Let  $q$  be the pseudo-quadratic form associated with the hermitian form used to defined  $\mathrm{SU}_3$  (see Remark 2.2). Explicitly, for  $x \in L^3$ ,  $q(x) = lf(x, x) + L^0$ , where  $L^0 = \{x \in L \mid x + \bar{x} = 0\}$  (see [BT72, 10.1.1 (7), (8)]). For  $x \in L$ , we define  $\omega_q(x) = \frac{1}{2} \sup\{\omega(k) \mid k \in q((0, x, 0))\} = \frac{1}{2} \sup\{\omega(k) \mid k \in l\bar{x}x + L^0\}$ .

We can actually compute explicitly the value of  $\omega_q$ .

**Lemma A.5.**

1.  $\omega_q(x) = \omega(x) + \omega_q(1)$
2.  $\omega_q(1) = \frac{1}{2}\omega(l)$

Hence,  $\omega_q(x) = \omega(x) + \frac{1}{2}\omega(l)$

*Proof.* The first property follows from the definition, and the second one is Lemma A.3.  $\square$

**Definition A.6** ([BT72, 10.1.27]). Let  $\{e_{-1}, e_0, e_1\}$  be the canonical basis of  $L^3$ . For  $g \in \mathrm{End}(L^3)$ , let  $(g_{ij})$  be the matrix of  $g$  in the basis  $\{e_{-1}, e_0, e_1\}$ . For  $i, j \in \{-1, 0, 1\}$ , we define  $\bar{\omega}_{ij}(g) = \tilde{\omega}_i(g_{ij}) - \tilde{\omega}_j(1)$ , where  $\tilde{\omega}_{\pm 1} = \omega$ , while  $\tilde{\omega}_0 = \omega_q$ .

**Remark A.7.** One readily check that this definition agrees with the one given in [BT72, 10.1.27]. Indeed, we can take advantage of the fact that  $X_0$  is one dimensional. Let us identify  $\mathrm{Hom}(X_j, X_i)$  with  $L$ , through the basis  $\{e_{-1}, e_0, e_1\}$ , and define  $\omega_i$  as in [BT72, 10.1.27]. Then, for  $x \in L$  and  $\alpha \in \mathrm{Hom}(X_j, X_i) \cong L$ , we have  $\omega_i(\alpha(xe_j)) - \omega_j(xe_j) = \omega_i((\alpha x)e_i) - \omega_j(xe_j) = \tilde{\omega}_i(\alpha x) - \tilde{\omega}_j(x) = \tilde{\omega}_i(\alpha) - \tilde{\omega}_j(1)$ .

**Definition A.8** ([BT72, Corollaire 10.1.33]). With these notations,  $\hat{P}_x = \{g \in \mathrm{SU}_3(K) \mid \bar{\omega}_{ij}(g) \geq a_i(x) - a_j(x), i, j \in \{-1, 0, 1\}\}$ .

Note that we can omit the factor  $\frac{1}{2}\omega c(g)$  appearing in loc. cit. since by definition,  $c(g)$  is the similitude ratio (see [BT72, Definition 10.1.4]) and is equal to 1 for  $g \in \mathrm{SU}_3$ .

Again, this description depends on the choice of a non-trivial element in  $\mathbf{R}^*$ . Now, if we choose  $a_1: \mathbf{R} \rightarrow \mathbf{R}: x \rightarrow \frac{x}{2}$ , then for  $x \in \mathbf{R}$ , the group  $\hat{P}_x$  of Definition A.8 is indeed equal to the group  $P_x$  of Definition 3.7. To end the comparison between [BT72, Definition 7.4.2] and our Definition 3.10, one has to show that the maps  $\nu: N \rightarrow \mathrm{Aff}(\mathbf{R})$  are the same. This is easily obtained by comparing [BT72, Proposition 10.1.28 (iii)] with our Definition 3.9.

## B A review of the theory of CSA over local fields

Let  $D$  be a central division algebra of degree  $d$  over a local field  $K$  (recall that the degree of  $D$  over  $K$  is the square root of the dimension of the  $K$ -vector space  $D$ ). It is well known that such division algebras are classified (up to isomorphism) by elements of  $(\mathbf{Z}/d\mathbf{Z})^\times$ .

To be explicit, for  $r \in (\mathbf{Z}/d\mathbf{Z})^\times$ , the corresponding division algebra is the cyclic algebra  $(E/K, \sigma^r, \pi_K)$  where

- $E$  is the unramified extension of  $K$  of dimension  $d$ .
- $\sigma \in \mathrm{Gal}(E/K)$  is the element in  $\mathrm{Gal}(E/K)$  inducing the Frobenius automorphism on  $\bar{E}$ .

For the reader's convenience, we recall the definition of a cyclic algebra.

**Definition B.1.** Let  $K$  be a field and let  $E/K$  be a cyclic extension of degree  $d$ . Let  $\sigma$  be a generator of  $\mathrm{Gal}(E/K)$ , and let  $a \in K^\times$ . The cyclic algebra  $(E/K, \sigma, a)$  is defined as follows:

- $(E/K, \sigma, a) = \bigoplus_{i=0}^{d-1} u^i E$
- $u^{-1}xu = \sigma(x)$ , for all  $x \in E$

- $u^d = a$

**Definition B.2.** As in [dlST15], for a finite central division algebra  $D$  of degree  $d$  over  $K$ , we call the corresponding element  $r \in (\mathbf{Z}/d\mathbf{Z})^\times$  the Hasse invariant of  $D$ .

An important fact about such a division algebra  $D$  is that it splits over  $E$ . It is important for us to describe explicitly the embedding of  $D$  inside  $M_d(E)$ , the algebra of  $d \times d$  matrices with coefficients in  $E$ .

**Definition B.3.** Let  $D$  be a division algebra isomorphic to the cyclic algebra  $(E/K, \sigma^r, \pi_K)$  of degree  $d$  over  $K$ . Consider the isomorphism of (right)  $E$ -vector spaces

$$f: E^n \rightarrow D: v = (v_1, \dots, v_n) \mapsto \sum_{i=0}^{d-1} u^i v_{i+1}$$

Let  $\varphi: D \rightarrow M_n(E): x \mapsto (v \mapsto f^{-1}(x.f(v)))$ . More explicitly,

$$\varphi\left(\sum_{i=0}^{d-1} u^i x_{i+1}\right) = \begin{pmatrix} x_1 & \pi_K \sigma^r(x_d) & \pi_K \sigma^{2r}(x_{d-1}) & \dots & \pi_K \sigma^{(d-1)r}(x_2) \\ x_2 & \sigma^r(x_1) & \pi_K \sigma^{2r}(x_d) & \dots & \pi_K \sigma^{(d-1)r}(x_3) \\ x_3 & \sigma^r(x_2) & \sigma^{2r}(x_1) & \dots & \pi_K \sigma^{(d-1)r}(x_4) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_d & \sigma^r(x_{d-1}) & \sigma^{2r}(x_{d-1}) & \dots & \sigma^{(d-1)r}(x_1) \end{pmatrix}$$

We can now properly spell out the definition of the reduced norm.

**Definition B.4.** Let  $D$  be a division algebra isomorphic to the cyclic algebra  $(E/K, \sigma^r, \pi_K)$  of degree  $d$  over  $K$ . We define the reduced norm  $\text{Nrd}$  as follows:

$$\text{Nrd}: M_n(D) \rightarrow K: g \rightarrow \det(\varphi(g_{ij}))$$

where  $\varphi(g_{ij})$  is seen as a  $(dn)^2$  matrix with coefficient in  $E$

We end this discussion by an analysis of the ring of integers  $D$ .

**Lemma B.5.** Let  $D$  be a central division algebra over  $K$  of degree  $d$ , and let  $r \in \mathbf{N} \cup \{\infty\}$  (with the convention that  $\mathfrak{m}^\infty = (0)$ ). Since  $E$  is unramified,  $\mathcal{O}_E/\mathfrak{m}_E^r \cong \mathcal{O}_K/\mathfrak{m}_K^r \oplus \dots \oplus \mathcal{O}_K/\mathfrak{m}_K^r$ . Furthermore,  $\mathcal{O}_D/\mathfrak{m}_D^{rd} \cong \bigoplus_{i=0}^{d-1} u^i \mathcal{O}_E/\mathfrak{m}_E^r$ . Otherwise stated,  $\mathcal{O}_D/\mathfrak{m}_D^{rd}$  is a free  $\mathcal{O}_E/\mathfrak{m}_E^r$ -module, and we can define a map  $\bar{\varphi}: \mathcal{O}_D/\mathfrak{m}_D^{rd} \hookrightarrow M_d(\mathcal{O}_E/\mathfrak{m}_E^r)$ , which is compatible with the map  $\varphi$  of Definition B.3, in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_D & \hookrightarrow & M_d(\mathcal{O}_E) \\ \downarrow & & \downarrow \\ \mathcal{O}_D/\mathfrak{m}_D^{rd} & \hookrightarrow & M_d(\mathcal{O}_E/\mathfrak{m}_E^r) \end{array}$$

*Proof.* This is straightforward from the definitions. □

## C An integral model of $\text{SL}_2(D)$

Recall that the group  $\text{SL}_2(D)$  consists of the  $2 \times 2$  matrices with coefficient in  $D$  having reduced norm 1 (Definition 2.4). Recall the definition of the embedding  $\varphi: D \rightarrow M_n(E)$  given in Definition B.3. In view of the definition of the reduced norm (Definition B.4), we arrive at the following explicit definition of  $\text{SL}_2(D)$ .

**Definition C.1.**  $\text{SL}_2(D) = \{ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \mid g_{ij} \in D, \det(\varphi(g_{ij})) = 1 \}$

Mimicking this definition, we can define a similar group over  $\mathcal{O}_D/\mathfrak{m}_D^{rd}$ .

**Definition C.2.** Let  $D$  be a central division algebra over  $K$  of degree  $d$ , and let  $r \in \mathbf{N} \cup \{\infty\}$ . Keeping the notations of Lemma B.5, we define

$$\mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) = \left\{ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \mid g_{ij} \in \mathcal{O}_D/\mathfrak{m}_D^{rd}, \det \left( \begin{pmatrix} \overline{\varphi}(g_{11}) & \overline{\varphi}(g_{12}) \\ \overline{\varphi}(g_{21}) & \overline{\varphi}(g_{22}) \end{pmatrix} \right) = 1 \right\}$$

Let us now discuss the underlying algebraic group of  $\mathrm{SL}_2(D)$ . Let  $M_2(D)$  be the algebra of  $2 \times 2$  matrices with coefficient in  $D$ . Using the embedding  $D \hookrightarrow M_d(E)$ , we can identify  $M_2(D)$  with a  $K$ -linear subspace of  $M_{2d}(E)$ . Now,  $\mathrm{SL}_2(D)$  is the closed subspace of  $M_2(D) \cong \mathbf{A}_K^{(2d)^2}$  cut out by the polynomial equation  $\mathrm{Nrd} = 1$ . We can now mimic this situation over the ring of integers to define an integral model of  $\mathrm{SL}_2(D)$ .

**Definition C.3.** Let  $D$  be a central division algebra of degree  $d$  over  $K$ , and let  $M_2(\mathcal{O}_D)$  be the  $\mathcal{O}_K$ -algebra of  $2 \times 2$  matrices with coefficient in  $\mathcal{O}_D$ . Using the embedding  $\mathcal{O}_D \hookrightarrow M_d(\mathcal{O}_E)$ , where  $E$  is the unramified extension of  $K$  of degree  $d$ , we can identify  $M_2(\mathcal{O}_D)$  with a free  $\mathcal{O}_K$ -submodule of  $M_{2d}(\mathcal{O}_E)$ . We define the  $\mathcal{O}_K$ -scheme  $\underline{\mathrm{SL}}_{2,D}$  to be the closed subscheme of  $M_2(\mathcal{O}_D) \cong \mathbf{A}_{\mathcal{O}_K}^{(2d)^2}$  cut out by the polynomial equation  $\mathrm{Nrd} = 1$ .

Of course, the crucial point is to check that  $\underline{\mathrm{SL}}_{2,D}$  is in fact smooth.

**Theorem C.4.**  $\underline{\mathrm{SL}}_{2,D}$  is a smooth  $\mathcal{O}_K$ -scheme.

*Proof.* This is one of the main results in [BT84b]. Let us explain how to extract it from there. Let  $\varphi$  be the valuation of  $GL_2(D)$  defined in [BT84b, 2.2, display (4)]. The valuation  $\varphi$  is thus a point of the enlarged apartment  $A_1$ . The associated norm is defined as  $\alpha_\varphi(e_1x_1 + e_2x_2) = \inf\{\omega(x_1), \omega(x_2)\}$  (following the definition in [BT84b, 2.8, display (9)]). The corresponding order  $\mathcal{M}_{\alpha_\varphi}$  of  $M_2(D)$  defined in [BT84b, 1.17] is  $\{ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in M_2(D) \mid \omega(g_{ij}) \geq 0 \}$  (this is easily computed using the description of  $\mathrm{End} \alpha(u)$  in [BT84b, 1.11, display (17)]). Note that  $\mathcal{M}_{\alpha_\varphi}$  is isomorphic to the affine space  $\mathbf{A}_{\mathcal{O}_K}^{(2d)^2}$  (being a free  $\mathcal{O}_K$ -module). Finally, following [BT84b, 3.6], let  $\mathfrak{G}_\varphi$  be the (principal) open subscheme of the affine space  $\mathcal{M}_{\alpha_\varphi}$  defined by the non-vanishing of the reduced norm (see also [BT84b, 3.2]).

$\mathfrak{G}_\varphi$  is actually an integral model for  $GL_2(D)$ , and the  $\mathrm{SL}_2(D)$  case is then treated in [BT84b, §5]. Let  $\mathfrak{G}_{1,\varphi}$  be the schematic adherence of  $\mathrm{SL}_2(D)$  in  $\mathfrak{G}_\varphi$  (following the definition in [BT84b, 5.3]). It is mentioned in [BT84b, 5.5] that the group  $\mathfrak{G}_{1,\varphi}$  is the closed subgroup of  $\mathfrak{G}_\varphi$  defined by the equation  $\mathrm{Nrd} = 1$ , and hence it coincides with our group  $\underline{\mathrm{SL}}_{2,D}$ . But by [BT84b, 5.5],  $\mathfrak{G}_{1,\varphi}$  is smooth over  $\mathcal{O}_K$ , concluding the proof. Note that to apply [BT84b, 5.5], we should check that a finite unramified extension of a local field is étale in the sense of [BT84b]. But this is clear in view of [BT84a, 1.6.1 (f) and Definition 1.6.2].  $\square$

We conclude our study of the  $\mathrm{SL}_2(D)$  case by identifying the rational points of  $\underline{\mathrm{SL}}_{2,D}$ .

**Lemma C.5.** Let  $D$  be a central division algebra over  $K$  of degree  $d$ , and let  $r \in \mathbf{N} \cup \{\infty\}$ . Then  $\underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K/\mathfrak{m}_K^r) \cong \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd})$  (where by convention,  $\mathfrak{m}^\infty = (0)$ ).

*Proof.* Because the diagram appearing in Lemma B.5 is commutative, we have

$$\begin{aligned} \underline{\mathrm{SL}}_{2,D}(D)(\mathcal{O}_K/\mathfrak{m}_K^r) &= \{ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in M_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) \mid \mathrm{Nrd}(g) = 1 \} \\ &\cong \{ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in M_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) \mid \det \left( \begin{pmatrix} \overline{\varphi}(g_{11}) & \overline{\varphi}(g_{12}) \\ \overline{\varphi}(g_{21}) & \overline{\varphi}(g_{22}) \end{pmatrix} \right) = 1 \} \end{aligned}$$

as wanted.  $\square$

# References

- [BT73] Armand Borel and Jacques Tits, *Homomorphismes “abstraites” de groupes algébriques simples*, Ann. of Math. (2) **97** (1973), 499–571 (French). MR0316587
- [BT72] François Bruhat and Jacques Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251 (French). MR0327923
- [BT84a] François Bruhat and Jacques Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. **60** (1984), 197–376 (French). MR756316
- [BT84b] François Bruhat and Jacques Tits, *Schémas en groupes et immeubles des groupes classiques sur un corps local*, Bull. Soc. Math. France **112** (1984), no. 2, 259–301 (French). MR788969
- [BT87] François Bruhat and Jacques Tits, *Schémas en groupes et immeubles des groupes classiques sur un corps local. II. Groupes unitaires*, Bull. Soc. Math. France **115** (1987), no. 2, 141–195 (French, with English summary). MR919421
- [BM96] Marc Burger and Shahar Mozes, *CAT(-1)-spaces, divergence groups and their commensurators*, J. Amer. Math. Soc. **9** (1996), no. 1, 57–93, DOI 10.1090/S0894-0347-96-00196-8.
- [CR16] Pierre-Emmanuel Caprace and Nicolas Radu, *Chabauty limits of simple groups acting on trees*, 2016. Preprint: <http://arxiv.org/abs/1608.00461>.
- [CS15] Pierre-Emmanuel Caprace and Thierry Stulemeijer, *Totally disconnected locally compact groups with a linear open subgroup*, Int. Math. Res. Not. IMRN **24** (2015), 13800–13829, DOI 10.1093/imrn/rnv086. MR3436164
- [Del84] Pierre Deligne, *Les corps locaux de caractéristique  $p$ , limites de corps locaux de caractéristique 0*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 119–157 (French). MR771673
- [DG70] Michel Demazure and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970 (French). Avec un appendice *Corps de classes local* par Michiel Hazewinkel. MR0302656
- [GE] Georges Elencwajg, *surjective map of rings with same dimension*, Mathematics Stack Exchange. URL: <http://math.stackexchange.com/q/604091> (version: 2013-12-12).
- [FV02] Ivan B. Fesenko and Sergei V. Vostokov, *Local fields and their extensions*, 2nd ed., Translations of Mathematical Monographs, vol. 121, American Mathematical Society, Providence, RI, 2002. With a foreword by Igor R. Shafarevich. MR1915966
- [Gro67] Alexandre Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 361 (French). MR0238860
- [Kaz86] David Kazhdan, *Representations of groups over close local fields*, J. Analyse Math. **47** (1986), 175–179, DOI 10.1007/BF02792537. MR874049
- [Lan94] Serge Lang, *Algebraic number theory*, 2nd ed., Graduate Texts in Mathematics, vol. 110, Springer-Verlag, New York, 1994. MR1282723
- [KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits. MR1632779
- [MS20] Stefan Mazurkiewicz and Waław Sierpiński, *Contribution à la topologie des ensembles dénombrables*, Fundamenta Mathematicae **1** (1920), no. 1, 17–27 (fre).
- [Pin98] Richard Pink, *Compact subgroups of linear algebraic groups*, J. Algebra **206** (1998), no. 2, 438–504, DOI 10.1006/jabr.1998.7439. MR1637068
- [Rad15] Nicolas Radu, *A classification theorem for boundary 2-transitive automorphism groups of trees*, 2015. Preprint (to appear in Invent. Math.): <http://arxiv.org/abs/1509.04913>.
- [dlST15] Mikael de la Salle and Romain Tessera, *Local-to-global rigidity of Bruhat-Tits buildings* (2015). Preprint: <http://arxiv.org/abs/1512.02775>.
- [TS16] The Stacks Project Authors, *Stacks Project*, 2016. URL: <http://stacks.math.columbia.edu>.
- [Stu] Thierry Stulemeijer, *Reference for Hensel’s Lemma in Algebraic Geometry*. URL: <http://mathoverflow.net/q/234709> (version: 2016-03-28).
- [Tit66] Jacques Tits, *Classification of algebraic semisimple groups*, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 33–62. MR0224710
- [Tit74] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin-New York, 1974. MR0470099

- [Tit79] Jacques Tits, *Reductive groups over local fields*, Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69. MR546588
- [Tit86] Jacques Tits, *Immeubles de type affine*, Buildings and the geometry of diagrams (Como, 1984), Lecture Notes in Math., vol. 1181, Springer, Berlin, 1986, pp. 159–190, DOI 10.1007/BFb0075514, (to appear in print) (French). MR843391
- [Wei09] Richard M. Weiss, *The structure of affine buildings*, Annals of Mathematics Studies, vol. 168, Princeton University Press, Princeton, NJ, 2009. MR2468338